

TOWARD AN ENUMERATIVE GEOMETRY WITH QUADRATIC FORMS

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ABSTRACT. Using the motivic stable homotopy category over a field k , a smooth projective variety X over k has an Euler characteristic $\chi^{cat}(X)$ in the Grothendieck-Witt ring $\mathrm{GW}(k)$. The rank of $\chi^{cat}(X)$ is the classical \mathbb{Z} -valued Euler characteristic, defined as the degree of the top Chern class of the tangent bundle of X .

The search for Grothendieck-Witt valued “Riemann-Hurwitz” formulas relating a global invariant of a smooth projective total space X with local invariants associated to the singularities of a map $f : X \rightarrow C$ to a curve was initiated by Kass and Wickelgren.

We develop tools to compute $\chi^{cat}(X)$, assuming k has characteristic $\neq 2$. and apply these to refine some classical formulas in enumerative geometry, such as the Riemann-Hurwitz formula, to identities in $\mathrm{GW}(k)$; these formulas are closely related to those obtained by Kass and Wickelgren. We also compute $\chi^{cat}(X)$ for all odd dimensional X , as well as for all hypersurfaces in \mathbb{P}_k^{n+1} defined by an equation of the form $\sum_{i=0}^{n+1} a_i X_i^m$; this latter includes the case of an arbitrary quadric hypersurface.

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INTRODUCTION

We work throughout in the category of smooth quasi-projective schemes over a field k , \mathbf{Sm}/k , with $\text{char}(k) \neq 2$. The main goal of this paper is to take steps toward constructing a good theory of enumerative geometry with values in quadratic forms, refining the classical \mathbb{Z} -valued enumerative geometry. The foundations of this theory have been laid by work of Barge-Morel [11], Fasel [12] and Morel [25, 26] (and many others), and first steps in this direction have been taken by Hoyois [17] and Kass-Wickelgren [21].

The main tool is the replacement of the Chow groups $\text{CH}^n(X)$ of a smooth variety X , viewed via Bloch's formula as the cohomology of the Milnor K -sheaves

$$\text{CH}^n(X) \cong H^n(X, \mathcal{K}_n^M),$$

with the *oriented Chow groups* of Barge-Morel [11, 12]

$$\widetilde{\text{CH}}^n(X; L) := H^n(X, \mathcal{K}_n^{MW}(L)).$$

Here $\mathcal{K}_n^{MW}(L)$ is the n th Milnor-Witt sheaf, as defined by Hopkins-Morel [25, 26], twisted by a line bundle L on X . This theory has many of the formal properties of the Chow ring, with the subtlety that the pushforward maps for a proper morphism $f : Y \rightarrow X$ of relative dimension d , are induced by the map

$$H^a(Y, \mathcal{K}_b^{MW}(f^*L \otimes \omega_{Y/k})) \rightarrow H^{a-d}(X, \mathcal{K}_{b-d}^{MW}(L \otimes \omega_{X/k})).$$

The second important difference is that, although a rank r vector bundle $V \rightarrow X$ has an Euler class [11, §2.1]

$$e(V) \in H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)),$$

the group this class lives in depends on V (or at least $\det V$). Under the map

$$\mathcal{K}_*^{MW}(\det^{-1} V) \rightarrow \mathcal{K}_*^M$$

$e(V)$ maps to the top Chern class $c_r(V)$ and $e(\det V)$ maps to $c_1(V)$, but there is no projective bundle formula for the oriented Chow groups, and thus no obvious “intermediate” Euler classes lifting the other Chern classes of V to the oriented setting.

There is still enough here to define an Euler characteristic of a smooth projective k -scheme $p : X \rightarrow \text{Spec } k$ as

$$\chi^{CW}(X) := p_*(e(T_X)) \in K_0^{MW}(k),$$

where, if X has dimension d over k , $e(T_X) \in H^d(X, \mathcal{K}_d^{MW}(\omega_{X/k}))$ is the Euler class of the tangent bundle T_X . Morel's theorem [25] identifies $K_0^{MW}(k)$ with the Grothendieck-Witt group of non-degenerate quadratic forms over k , $\text{GW}(k)$, so we have the Euler characteristic $\chi^{CW}(X) \in \text{GW}(k)$. The comparison of $e(T_X)$ with $c_{\text{top}}(T_X)$ shows that the image $\chi^{CW}(X)$ under the rank homomorphism $\text{GW}(k) \rightarrow \mathbb{Z}$ is the classical Euler characteristic of X , which agrees with the topological Euler characteristic of $X(\mathbb{C})$ defined

using singular cohomology, if $k \subset \mathbb{C}$, or the ℓ -adic Euler characteristic of $X_{\bar{k}}$, defined using étale cohomology.

One can also define a categorical Euler characteristic $\chi^{cat}(X)$, by using the infinite suspension spectrum $\Sigma_T^\infty X_+ \in \mathrm{SH}(k)$, where $\mathrm{SH}(k)$ is the motivic stable homotopy category over k . Hoyois [18, Theorem 5.22], Hu [19, Appendix A], Riou [30] and Voevodsky [37, §2] have shown that this suspension spectrum is always a dualizable object in $\mathrm{SH}(k)$, so it gives rise in a standard way to a canonical endomorphism of the unit object \mathbb{S}_k :

$$\chi^{cat}(X) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k).$$

By Morel's theorem [25, Theorem 6.4.1] there is a canonical isomorphism $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) \cong \mathrm{GW}(k)$, so we have a second Euler characteristic in $\mathrm{GW}(k)$.

We should mention that for $k \subset \mathbb{R} \subset \mathbb{C}$, the image of the categorical Euler characteristic $\chi^{cat}(X)$ in $\mathrm{GW}(\mathbb{R})$ has the property that its signature gives the Euler characteristic of $X(\mathbb{R})$, while as we have mentioned above, the rank gives the Euler characteristic of $X(\mathbb{C})$.

Our first main goal is the equality of the two classes.

Theorem 1. *Let X be a smooth projective variety of pure dimension over k . Then*

$$\chi^{CW}(X) = \chi^{cat}(X)$$

in $\mathrm{GW}(k)$.

This requires understanding the relationship between the Euler class as defined by Fasel, and analogous constructions within the motivic stable homotopy category. This in turn is based on understanding the motivic constructions of pull-back and pushforward on twisted Milnor-Witt cohomology. These comparisons occupy the first main part of the paper.

After proving Theorem 1, we derive some consequences, notably, the fact that the Euler characteristic of an odd dimensional smooth projective variety is always hyperbolic (Theorem 7.1); one can view this as a generalization of the fact that the topological Euler characteristic of a real oriented manifold of dimension $4m + 2$ is always even. We then turn to developing some computational techniques, first for line bundles and then for bundles of higher rank. Here the main goal is to compare $e(V)$ and $e(V \otimes L)$ for a line bundle L , without having the “lower Chern classes” of V on hand. We also prove a useful formula relating Euler class of a vector bundle V with that of its dual (Theorem 11.1):

$$e(V) = (-\langle -1 \rangle)^{\mathrm{rank} V} e(V^\vee),$$

where $\langle -1 \rangle$ is the one-dimensional form $q(x) = -x^2$. The proof of this uses the interpretation by Asok-Fasel [4] of the Euler class as an obstruction class.

Kass-Wickelgren¹ have constructed an Euler number in $\mathrm{GW}(k)$ for a relatively oriented algebraic vector bundle with enough sections on a smooth projective k -scheme; as one application, they use this to lift the count of

¹private communications, June, 2016 and March, 2017

lines on a smooth cubic surface over k to an equality in $\mathrm{GW}(k)$. For a pencil $f : X \rightarrow \mathbb{P}^1$ of curves on a smooth projective surface X over k , they lift the classical computation of the Euler number of $T_X^* \otimes f^* T_{\mathbb{P}^1}$ in terms of the singularities of the fibers of f to an equality in $\mathrm{GW}(k)$.

We approach the question of lifting such classical degeneration formulas to $\mathrm{GW}(k)$ from a somewhat different point of view. We apply Theorem 1 and the results obtained in §9-11 to give a generalization of the classical degeneration formulas for counting singularities in a morphism $f : X \rightarrow C$, with X a smooth projective variety and C a smooth projective curve (admitting for technical reasons a half-canonical line bundle); for X a curve, this a refinement of the classical Riemann-Hurwitz formula. Our generalization gives an identity in $\mathrm{GW}(k)$; applying the rank homomorphism recovers the classical numerical formulas. In the case of even dimensional varieties, we apply the degeneration formula to compute the Euler characteristic of smooth diagonalizable hypersurfaces, that is, a hypersurface $X \subset \mathbb{P}_k^{n+1}$ defined by an equation of the form $\sum_{i=0}^{n+1} a_i X_i^m$, see Theorem 13.1. As a special case, we find an explicit formula for the Euler characteristic of a quadric hypersurface, Corollary 13.2. The question of computing the Euler characteristic of a quadric hypersurface was raised by Kass-Wickelgren.

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1. THE CATEGORICAL EULER CHARACTERISTIC

Throughout this paper, we will work over a base-field k of characteristic $\neq 2$. It is often inconvenient to suppose that k is perfect, but as some basic results are not available in the non-perfect case, we will get around this technical difficulty by inverting the characteristic whenever working in the non-perfect case.

1.1. Duality for smooth projective varieties. We begin by recalling the construction of $\Sigma_T^\infty X_+^\vee$ by Hoyois, Hu, Riou and Voevodsky [18, 19, 30, 37]; our discussion follows [37, §2]. We first take the case $X = \mathbb{P}_k^d$, which we denote by \mathbb{P}^d ; unless mentioned otherwise, all products are over k .

Let $p_1, p_2 : \mathbb{P}^d \times \mathbb{P}^d \rightarrow \mathbb{P}^d$ be the projections. Let $p : \tilde{\mathbb{P}}^d \rightarrow \mathbb{P}^d$ be the Jouanolou cover of \mathbb{P}^d , that is, $\tilde{\mathbb{P}}^d$ is the open subscheme $\mathbb{P}^d \times \mathbb{P}^d \setminus H$ of $\mathbb{P}^d \times \mathbb{P}^d$, where H is the smooth divisor defined by the section $\sum_{i=0}^d X_i Y_i$ of $\mathcal{O}_{\mathbb{P}^d \times \mathbb{P}^d}(1, 1)$, and p is the restriction of the projection p_1 . $p : \tilde{\mathbb{P}}^d \rightarrow \mathbb{P}^d$ is thus an \mathbb{A}^d -bundle. Let $j : \tilde{\mathbb{P}}^d \rightarrow \mathbb{P}^d \times \mathbb{P}^d$ be the inclusion.

Let $i : \mathbb{P}^d \times \mathbb{P}^d \rightarrow \mathbb{P}^{d^2+2d}$ be the Segre embedding, which is defined by a choice of basis for $H^0(\mathbb{P}^d \times \mathbb{P}^d, \mathcal{O}(1, 1))$. Thus $H = i^*(H_\infty)$ for a suitable hyperplane H_∞ in \mathbb{P}^{d^2+d} and i restricts to a closed embedding $\tilde{i} : \tilde{\mathbb{P}}^d \rightarrow \mathbb{P}^{d^2+2d} \setminus H_\infty \cong \mathbb{A}^{d^2+2d}$. In particular, $\tilde{\mathbb{P}}^d$ is an affine scheme.

Let $p' : \tilde{\mathbb{P}}^d \rightarrow \mathbb{P}^d$ be the restriction of p_2 , let $\pi_1 : \tilde{\mathbb{P}}^d \times \mathbb{P}^d \rightarrow \tilde{\mathbb{P}}^d$, $\pi_2 : \tilde{\mathbb{P}}^d \times \mathbb{P}^d \rightarrow \mathbb{P}^d$ be the projections, and let $s : \tilde{\mathbb{P}}^d \rightarrow \tilde{\mathbb{P}}^d \times \mathbb{P}^d$ be the section

defined by the graph of p' . Let N be the normal bundle to i , \tilde{N} the restriction of N to $\tilde{\mathbb{P}}^d$, and $E \rightarrow \tilde{\mathbb{P}}^d$ the normal bundle to s .

Since $s = (\text{Id}, p')$, we have the canonical isomorphism $s^* \pi_2^* T_{\mathbb{P}^d} \cong p'^* T_{\mathbb{P}^d}$. The canonical isomorphism $T_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d} \cong \pi_1^* T_{\tilde{\mathbb{P}}^d} \oplus \pi_2^* T_{\mathbb{P}^d}$ induces the inclusion $i_2 : p'^* T_{\mathbb{P}^d} \rightarrow s^* T_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d}$. Composing i_2 with the surjection $\pi : s^* T_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d} \rightarrow E$ gives the isomorphism $p'^* T_{\mathbb{P}^d} \cong E$. This in turn induces isomorphisms

$$p^* T_{\mathbb{P}^d} \oplus E \cong p^* T_{\mathbb{P}^d} \oplus p'^* T_{\mathbb{P}^d} = j^* T_{\mathbb{P}^d \times \mathbb{P}^d} = T_{\tilde{\mathbb{P}}^d}.$$

Since $\tilde{\mathbb{P}}^d$ is affine, the exact sequence

$$(1.1) \quad 0 \rightarrow T_{\tilde{\mathbb{P}}^d} \rightarrow \tilde{i}^*(T_{\mathbb{A}^{d^2+d}}) \rightarrow \tilde{N} \rightarrow 0$$

splits, inducing an isomorphism $T_{\tilde{\mathbb{P}}^d} \oplus \tilde{N} \cong \tilde{i}^*(T_{\mathbb{A}^{d^2+d}})$, unique up to the choice of splitting. Composing with the pullback of the canonical isomorphism $T_{\mathbb{A}^{d^2+d}} \cong \mathcal{O}_{\mathbb{A}^{d^2+d}}^{d^2+2d}$ defines an isomorphism

$$T_{\tilde{\mathbb{P}}^d} \oplus \tilde{N} \cong \mathcal{O}_{\tilde{\mathbb{P}}^d}^{d^2+2d}.$$

Setting

$$\tilde{\nu}_{\tilde{\mathbb{P}}^d} := E \oplus \tilde{N},$$

this gives us an isomorphism

$$(1.2) \quad p^* T_{\mathbb{P}^d} \oplus \tilde{\nu}_{\tilde{\mathbb{P}}^d} \cong \mathcal{O}_{\tilde{\mathbb{P}}^d}^{d^2+2d},$$

canonical up to the choice of splitting in (1.1). Moreover, (1.2) gives a canonical isomorphism

$$(1.3) \quad \det \tilde{\nu}_{\tilde{\mathbb{P}}^d} \cong p^* \omega_{\mathbb{P}^d/k}$$

independent of the choice of splitting mentioned above.

Let $h = p \times \text{Id} : \tilde{\mathbb{P}}^d \times \mathbb{P}^d \rightarrow \mathbb{P}^d \times \mathbb{P}^d$ and let $\tilde{H} = h^{-1}(H)$. By construction, $\tilde{H} \cap s(\tilde{\mathbb{P}}^d) = \emptyset$ and thus for each $x \in \tilde{\mathbb{P}}^d$, the projective space $x \times_k \mathbb{P}^d \subset \tilde{\mathbb{P}}^d \times \mathbb{P}^d$ is endowed with the $k(x)$ -point $s(x)$ and the hyperplane $H_x := x \times \mathbb{P}^d \cap \tilde{H}$, with $s(x) \cap H_x = \emptyset$. Thus the linear projection $\ell_x : x \times \mathbb{P}^d \setminus s(x) \rightarrow H_x$ makes $x \times \mathbb{P}^d \setminus s(x)$ into a line bundle over H_x , and thus we have the morphism

$$\ell : \tilde{\mathbb{P}}^d \times \mathbb{P}^d \setminus s(\tilde{\mathbb{P}}^d) \rightarrow \tilde{H}$$

making $\tilde{\mathbb{P}}^d \times \mathbb{P}^d \setminus s(\tilde{\mathbb{P}}^d)$ into a line bundle over \tilde{H} with zero-section the inclusion $i_{\tilde{H}} : \tilde{H} \hookrightarrow \tilde{\mathbb{P}}^d \times \mathbb{P}^d \setminus s(\tilde{\mathbb{P}}^d)$. In particular, $i_{\tilde{H}}$ is an \mathbb{A}^1 -homotopy equivalence, with homotopy inverse ℓ .

Let

$$F = \mathbb{P}^{d^2+2d} / (H_\infty \cup (\mathbb{P}^{d^2+2d} \setminus i(\mathbb{P}^d \times \mathbb{P}^d)))$$

We have the evident map $T^{d^2+2d} \cong \mathbb{P}^{d^2+2d} / H_\infty \rightarrow F$. Voevodsky constructs an isomorphism of F with $\text{Th}(E \oplus \tilde{N})$ in $\mathcal{H}_\bullet(k)$ as follows:

Let N_H be the restriction of N to H . From the Morel-Voevodsky purity theorem, we have canonical isomorphisms (in $\mathcal{H}_\bullet(k)$)

$$\text{Th}_{\mathbb{P}^d \times \mathbb{P}^d}(N) \cong \mathbb{P}^{d^2+2d} / (\mathbb{P}^{d^2+2d} \setminus i(\mathbb{P}^d \times \mathbb{P}^d))$$

and

$$\mathrm{Th}_H(N_H) \cong H_\infty / (H_\infty \setminus H)$$

which yields the isomorphism

$$F \cong \mathrm{Th}_{\mathbb{P}^d \times \mathbb{P}^d}(N) / \mathrm{Th}_H(N_H).$$

The \mathbb{A}^1 -weak equivalence h induces an \mathbb{A}^1 -weak equivalence

$$\mathrm{Th}_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d}(h^*(N)) / \mathrm{Th}_{\tilde{H}}(h^*(N_H)) \rightarrow \mathrm{Th}_{\mathbb{P}^d \times \mathbb{P}^d}(N) / \mathrm{Th}_H(N_H).$$

A Mayer-Vietoris argument shows that the \mathbb{A}^1 -weak equivalence $i_{\tilde{H}} : \tilde{H} \rightarrow \tilde{\mathbb{P}}^d \times \mathbb{P}^d \setminus s(\tilde{\mathbb{P}}^d)$ extends to an \mathbb{A}^1 -weak equivalence

$$\mathrm{Th}(i_{\tilde{H}}) : \mathrm{Th}_{\tilde{H}}(h^*(N_H)) \rightarrow \mathrm{Th}_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d \setminus s(\tilde{\mathbb{P}}^d)}(h^*(N)),$$

giving an \mathbb{A}^1 -weak equivalence

$$\mathrm{Th}_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d}(h^*(N)) / \mathrm{Th}_{\tilde{H}}(h^*(N_H)) \rightarrow \mathrm{Th}_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d}(h^*(N)) / \mathrm{Th}_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d \setminus s(\tilde{\mathbb{P}}^d)}(h^*(N)).$$

But $\mathrm{Th}_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d}(h^*(N)) / \mathrm{Th}_{\tilde{\mathbb{P}}^d \times \mathbb{P}^d \setminus s(\tilde{\mathbb{P}}^d)}(h^*(N))$ is isomorphic to $h^*(N) / (h^*(N) \setminus s(\tilde{\mathbb{P}}^d))$; using homotopy purity again, this in turn is isomorphic to $\mathrm{Th}_{\tilde{\mathbb{P}}^d}(E \oplus \tilde{N})$, since the normal bundle to s in $h^*(N)$ is $E \oplus \tilde{N}$.

Thus, we have the canonical map in $\mathcal{H}_\bullet(k)$

$$\tilde{\eta}_{\mathbb{P}^d} : T^{d^2+2d} \rightarrow \mathrm{Th}_{\tilde{\mathbb{P}}^d}(E \oplus \tilde{N}) = \mathrm{Th}_{\tilde{\mathbb{P}}^d}(\tilde{\nu}_{\mathbb{P}^d}).$$

One defines

$$(\Sigma_T^\infty \mathbb{P}_+^d)^\vee := \Sigma_T^{-d^2-2d} \Sigma_T^\infty \mathrm{Th}_{\tilde{\mathbb{P}}^d}(\tilde{\nu}_{\mathbb{P}^d});$$

the map $\tilde{\eta}_{\mathbb{P}^d}$ thus gives the map in $\mathrm{SH}(k)$

$$\eta_{\mathbb{P}^d} : \mathbb{S}_k \rightarrow (\Sigma_T^\infty \mathbb{P}_+^d)^\vee.$$

Now consider a smooth quasi-projective k -scheme X of pure dimension n and take a locally closed embedding $i_X : X \hookrightarrow \mathbb{P}^d$ for some d . Let N_X be the normal bundle of X in \mathbb{P}^d , let $\tilde{X} = p^{-1}(X)$ with inclusion $i_{\tilde{X}} : \tilde{X} \rightarrow \tilde{\mathbb{P}}^d$ and projection $p_X : \tilde{X} \rightarrow X$. Let $\tilde{N}_X = p_X^*(N_X)$, so \tilde{N}_X is the normal bundle of \tilde{X} in $\tilde{\mathbb{P}}^d$. We define the vector bundle $\tilde{\nu}_{\tilde{X}}$ on \tilde{X} by

$$(1.4) \quad \tilde{\nu}_{\tilde{X}} := \tilde{N}_X \oplus i_{\tilde{X}}^* \tilde{\nu}_{\mathbb{P}^d}.$$

Splitting the pull-back to \tilde{X} of the exact sequence on X

$$0 \rightarrow T_X \rightarrow i_X^* T_{\mathbb{P}^d} \rightarrow N_X \rightarrow 0$$

gives an isomorphism

$$(1.5) \quad p_X^*(T_X) \oplus \tilde{N}_X \cong i_{\tilde{X}}^*(p^* T_{\mathbb{P}^d}),$$

unique up to the choice of the splitting. This together with the isomorphism (1.2) induces the isomorphism

$$(1.6) \quad p_X^*(T_X) \oplus \tilde{\nu}_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}^{d^2+2d},$$

again, unique up to the choice of splittings. As above, this induces a canonical isomorphism

$$(1.7) \quad \det \tilde{\nu}_{\tilde{X}} \cong p_X^* \omega_{X/k}.$$

Definition 1.1. We let $\Sigma_T^\infty X_+^{\tilde{\vee}}$ denote the shifted suspension spectrum $\Sigma_T^{-d^2-2d} \Sigma_T^\infty \mathrm{Th}(\nu_{\tilde{X}})$.

Remark 1.2. As defined, $\Sigma_T^\infty X_+^{\tilde{\vee}}$ depends on the given embedding $X \subset \mathbb{P}^d$. In fact, the object $\Sigma_T^{-d^2-2d} \Sigma_T^\infty \mathrm{Th}(\tilde{\nu}_{\tilde{X}})$ has a natural description in $\mathrm{SH}(k)$.

Let $\pi : X \rightarrow \mathrm{Spec} k$ be the structure morphism, giving the functor $\pi_\# : \mathbf{Sm}/X \rightarrow \mathbf{Sm}/k$ sending $p : Y \rightarrow X \in \mathbf{Sm}/X$ to $\pi \circ p : Y \rightarrow \mathrm{Spec} k$. This extends to the functor $\pi_\# : \mathbf{Spc}_\bullet(X) \rightarrow \mathbf{Spc}_\bullet(k)$. For $V \rightarrow Y$ a vector bundle, we have the Thom space $\mathrm{Th}_X(V) \in \mathbf{Spc}_\bullet(X)$, defined as the cofiber of the inclusion $V \setminus 0_Y \rightarrow V$ in $\mathbf{Spc}(X)$, with $\pi_\#(\mathrm{Th}_X(V)) = \mathrm{Th}(\pi_\# V) \in \mathbf{Spc}_\bullet(k)$.

We have the T -suspension functors

$$\Sigma_T^\infty : \mathbf{Spc}_\bullet(X) \rightarrow \mathrm{SH}(X), \quad \Sigma_T^\infty : \mathbf{Spc}_\bullet(k) \rightarrow \mathrm{SH}(k),$$

via which $\pi_\#$ extends to an exact functor $\pi_\# : \mathrm{SH}(X) \rightarrow \mathrm{SH}(k)$. Define $\mathrm{Th}_X(-T_X) := \Sigma_T^{-d^2-2d} \Sigma_T^\infty(\mathrm{Th}_X(\tilde{\nu}_{\tilde{X}}))$. The isomorphism (1.6) gives rise to an isomorphism

$$\mathrm{Th}_X(\tilde{\nu}_{\tilde{X}}) \wedge_X \mathrm{Th}_X(T_X) \cong \Sigma_T^{d^2+2d} \mathbb{S}_X$$

in $\mathbf{Spc}_\bullet(X)$, where \wedge_X is the monoidal product in $\mathbf{Spc}_\bullet(X)$. This isomorphism is independent of the various splittings used to give the isomorphism (1.6) and yields a canonical isomorphism

$$\mathrm{Th}_X(-T_X) \wedge \mathrm{Th}_X(T_X) \cong \mathbb{S}_X$$

in $\mathrm{SH}(X)$, that is, $\mathrm{Th}_X(T_X)$ is an invertible object of $\mathrm{SH}(X)$ with inverse $\mathrm{Th}_X(-T_X)$.

It follows from Ayoub's definition of $\pi_!$ [9, §1.5.3] that the identification $\mathrm{Th}_X(-T_X) \cong \mathrm{Th}_X(T_X)^{-1}$ gives a canonical isomorphism

$$\pi_!(\mathbb{S}_X) \cong \pi_\#(\mathrm{Th}_X(-T_X)),$$

which in turn gives the canonical isomorphism

$$\pi_!(\mathbb{S}_X) \cong \Sigma_T^{-d^2-2d} \Sigma_T^\infty \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) = \Sigma_T^\infty X_+^{\tilde{\vee}}$$

in $\mathrm{SH}(k)$. In short, the Thom space $\mathrm{Th}(\tilde{\nu}_{\tilde{X}})$ together with the Jouanolou cover $p_X : \tilde{X} \rightarrow X$ and the isomorphism (1.6) (up to choice of splittings) determine the object $\Sigma_T^\infty X_+^{\tilde{\vee}}$ of $\mathrm{SH}(k)$ up to canonical isomorphism; in particular, $\Sigma_T^\infty X_+^{\tilde{\vee}}$ is independent, up to canonical isomorphism, of the choice of locally closed immersion $X \rightarrow \mathbb{P}_k^d$.

To avoid having to trace through numerous diagrams to show that the various maps we will define below are in fact well-defined and satisfy the relations we need, we will use the explicit model $\Sigma_T^\infty X_+^{\tilde{\vee}}$ for $\pi_!(\mathbb{S}_X)$ in the

ensuing discussion, and we will ignore the question of whether our constructions are independent of the choice of locally closed immersion. In case X is projective, the identification of $\Sigma_T^\infty X_+^{\tilde{V}}$ with the dual of $\Sigma_T^\infty X_+$ via the maps δ_X and χ defined below rigidifies the situation without having to rely on the four-functor formalism.

We recall from [37, §2] the construction of the morphism

$$ev_X : \Sigma_T^\infty X_+^{\tilde{V}} \wedge \Sigma_T^\infty X_+ \rightarrow \mathbb{S}_k.$$

Let $q_1 : \tilde{X} \times X \rightarrow \tilde{X}$ be the projection, $\tilde{\Delta}_X^1 : \tilde{X} \rightarrow \tilde{X} \times X$ the section determined by the restriction $p_X : \tilde{X} \rightarrow X$ of p . Let $s_0^1 : \tilde{X} \times X \rightarrow q_1^*(\tilde{\nu}_{\tilde{X}})$ be the 0-section.

We have the map

$$\pi_1 : \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \wedge X_+ \rightarrow \mathrm{Th}(q_1^*(\tilde{\nu}_{\tilde{X}}))/[\mathrm{Th}(q_1^*(\tilde{\nu}_{\tilde{X}})) \setminus s_0^1 \circ \tilde{\Delta}_X(\tilde{X})]$$

defined as composition

$$\mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \wedge X_+ \cong \mathrm{Th}(q_1^*(\tilde{\nu}_{\tilde{X}})) \rightarrow \mathrm{Th}(q_1^*(\tilde{\nu}_{\tilde{X}}))/[\mathrm{Th}(q_1^*(\tilde{\nu}_{\tilde{X}})) \setminus s_0^1 \circ \tilde{\Delta}_X^1(\tilde{X})].$$

By homotopy purity, this last space is canonically isomorphic to the Thom space $\mathrm{Th}(N_{\tilde{\Delta}_X^1} \oplus \tilde{\nu}_{\tilde{X}})$, where $N_{\tilde{\Delta}_X^1}$ is the normal bundle of the section $\tilde{\Delta}_X^1$.

Letting $\Delta_X : X \rightarrow X \times X$ be the diagonal, we identify the normal bundle N_{Δ_X} with T_X via the composition

$$T_X = \Delta_X^* p_1^* T_X \xrightarrow{di_1} \Delta_X^* (p_1^* T_X \oplus p_2^* T_X) \cong \Delta_X^* T_{X \times X} \xrightarrow{\rho} N_{\Delta_X}$$

where i_1 is the inclusion as the first summand and ρ is the canonical surjection. Pulling back by p_X^* , this gives the isomorphism

$$(1.8) \quad p_X^* T_X \xrightarrow{di_1} N_{\tilde{\Delta}_X^1}.$$

Combining (1.8) and (1.6) gives the isomorphism

$$(1.9) \quad N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}} \xrightarrow{\alpha_1} i_{\tilde{X}}^*(p^* T_{\mathbb{P}^d} \oplus \tilde{\nu}_{\mathbb{P}^d}) \cong O_{\tilde{X}}^{d^2+2d},$$

inducing an isomorphism on the Thom spaces

$$\mathrm{Th}(N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}}) \cong \mathrm{Th}(O_{\tilde{X}}^{d^2+2d}) \cong \Sigma_T^{d^2+2d} \tilde{X}_+,$$

unique up to \mathbb{A}^1 -homotopy. The composition of these maps gives us the map

$$\epsilon_{X \subset \mathbb{P}^d} : \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \wedge X_+ \rightarrow \Sigma_T^{d^2+2d} \tilde{X}_+.$$

Composition $\epsilon_{X \subset \mathbb{P}^d}$ with the suspension of the structure morphism

$$\Sigma_T^{d^2+2d} \pi_{\tilde{X}} : \Sigma_T^{d^2+2d} \tilde{X}_+ \rightarrow \Sigma_T^{d^2+2d} \mathrm{Spec} k_+ = T^{d^2+2d}$$

defines the map

$$ev_{X \subset \mathbb{P}^d} : \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \wedge X_+ \rightarrow T^{d^2+2d}.$$

Stabilizing and desuspending gives the maps

$$\begin{aligned}\epsilon_X : (\Sigma_T^\infty X_+)^{\tilde{\vee}} \wedge \Sigma_T^\infty X_+ &\rightarrow \Sigma_T^\infty X_+ \\ ev_X : (\Sigma_T^\infty X_+)^{\tilde{\vee}} \wedge \Sigma_T^\infty X_+ &\rightarrow \mathbb{S}_k\end{aligned}$$

in $\mathrm{SH}(k)$.

Suppose now that X is projective, so $i_X(X)$ is closed in \mathbb{P}^d and $i_{\tilde{X}}(\tilde{X})$ is closed in $\tilde{\mathbb{P}}^d$. Let $s_0 : \tilde{\mathbb{P}}^d \rightarrow \tilde{\nu}_{\tilde{\mathbb{P}}^d}$ be the 0-section. Homotopy purity gives the isomorphism in $\mathcal{H}_\bullet(k)$

$$\mathrm{Th}(\tilde{\nu}_{\tilde{X}}) = \mathrm{Th}(\tilde{N}_X \oplus i_X^* \tilde{\nu}_{\tilde{\mathbb{P}}^d}) \cong \mathrm{Th}(\tilde{\nu}_{\tilde{\mathbb{P}}^d}) / [\mathrm{Th}(\tilde{\nu}_{\tilde{\mathbb{P}}^d} \setminus i_{\tilde{X}}(\tilde{X}))]$$

The quotient map

$$\pi_{X \subset \mathbb{P}^d} : \mathrm{Th}(\tilde{\nu}_{\tilde{\mathbb{P}}^d}) \rightarrow \mathrm{Th}(\tilde{\nu}_{\tilde{\mathbb{P}}^d}) / [\mathrm{Th}(\tilde{\nu}_{\tilde{\mathbb{P}}^d} \setminus i_{\tilde{X}}(\tilde{X}))]$$

composed with $\tilde{\eta}_{\tilde{\mathbb{P}}^d}$ thus gives the map

$$\eta_{X \subset \mathbb{P}^d} : T^{d^2+2d} \rightarrow \mathrm{Th}(\tilde{\nu}_{\tilde{X}})$$

in $\mathcal{H}_\bullet(k)$. Stabilizing and taking a desuspension, the map $\tilde{\eta}_{X \subset \mathbb{P}^d}$ yields the map

$$\eta_X : \mathbb{S}_k \rightarrow (\Sigma_T^\infty X_+)^{\tilde{\vee}}$$

in $\mathrm{SH}(k)$.

Let $q_2 : X \times \tilde{X} \rightarrow \tilde{X}$ be the projection and $\tilde{\Delta}_X^2 : \tilde{X} \rightarrow X \times \tilde{X}$ the section to q_2 defined by the morphism p_X . $\tilde{\Delta}_X^2$ extends to a bundle map $\tilde{\Delta}_X^2 : \tilde{\nu}_{\tilde{X}} \rightarrow q_2^*(\tilde{\nu}_{\tilde{X}})$ over $\tilde{\Delta}_X$, giving the map on Thom spaces

$$\mathrm{Th}(\tilde{\Delta}_X^2) : \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \rightarrow \mathrm{Th}(q_2^*(\tilde{\nu}_{\tilde{X}})).$$

Composed with the canonical isomorphism in $\mathbf{Spc}_\bullet(k)$

$$\mathrm{Th}(q_2^*(\tilde{\nu}_{\tilde{X}})) \cong X_+ \wedge \mathrm{Th}(\tilde{\nu}_{\tilde{X}})$$

we have the extension of $\tilde{\Delta}_X^2$ to a map

$$\mathrm{Th}(\tilde{\Delta}_X^2) : \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \rightarrow X_+ \wedge \mathrm{Th}(\tilde{\nu}_{\tilde{X}}).$$

Define the map

$$\delta_{X \subset \mathbb{P}^d} : T^{d^2+2d} \rightarrow X_+ \wedge \mathrm{Th}(\tilde{\nu}_{\tilde{X}})$$

by

$$\delta_{X \subset \mathbb{P}^d} := \mathrm{Th}(\tilde{\Delta}_X^2) \circ \tilde{\eta}_{X \subset \mathbb{P}^d}.$$

Stabilizing and desuspending gives the map

$$\delta_X : \mathbb{S}_k \rightarrow \Sigma_T^\infty X_+ \wedge (\Sigma_T^\infty X_+)^{\tilde{\vee}}.$$

We refer the reader to §1.2 for the the relevant notions of duality.

Proposition 1.3. *The triple $(\Sigma_T^\infty X_+^{\tilde{\vee}}, \delta_X, ev_X)$ is the dual of $\Sigma_T^\infty X_+$ in $\mathrm{SH}(k)$.*

Proof. We need to verify the identities

$$(\mathrm{Id}_{\Sigma_T^\infty X_+} \wedge ev_X) \circ (\delta_X \wedge \mathrm{Id}_{\Sigma_T^\infty X_+}) = \mathrm{Id}_{\Sigma_T^\infty X_+}$$

and

$$(ev_X \wedge \mathrm{Id}_{\Sigma_T^\infty X_+^\vee}) \circ (\mathrm{Id}_{\Sigma_T^\infty X_+^\vee} \wedge \delta_X) = \mathrm{Id}_{\Sigma_T^\infty X_+^\vee}$$

in $\mathrm{SH}(k)$. We give the proof of the first identity; the argument for the second one is similar and is left to the reader.

Choosing an embedding $i_X : X \rightarrow \mathbb{P}^d$ and letting $M = d^2 + 2d$, the first identity follows from the identity in $\mathcal{H}_\bullet(k)$

$$(1.10) \quad (\mathrm{Id}_{X_+} \wedge ev_{X \subset \mathbb{P}^d}) \circ (\delta_{X \subset \mathbb{P}^d} \wedge \mathrm{Id}_{X_+}) = \tau_{T^M, X_+} : T^M \wedge X_+ \rightarrow X_+ \wedge T^M.$$

We write $\tilde{\nu}$ for $\tilde{\nu}_{\tilde{X}}$. Letting

$$\begin{aligned} p_2^{123} : X \times \tilde{X} \times X &\rightarrow \tilde{X} \\ p_2^{23} : \tilde{X} \times X &\rightarrow \tilde{X} \end{aligned}$$

be the projections, we have the canonical isomorphisms

$$\begin{aligned} \mathrm{Th}(p_2^{123*} \tilde{\nu}) &\cong X_+ \wedge \mathrm{Th}(\tilde{\nu}) \wedge X_+ \\ \mathrm{Th}(p_2^{23*} \tilde{\nu}) &\cong \mathrm{Th}(\tilde{\nu}) \wedge X_+ \end{aligned}$$

in $\mathcal{H}_\bullet(k)$. We write $\delta \wedge \mathrm{Id}_{X_+}$ for the composition

$$T^M \wedge X_+ \xrightarrow{\delta_{X \subset \mathbb{P}^d} \wedge \mathrm{Id}_{X_+}} X_+ \wedge \mathrm{Th}(\tilde{\nu}) \wedge X_+ \cong \mathrm{Th}(p_2^{123*} \tilde{\nu})$$

and define morphisms $\eta \wedge \mathrm{Id}_{X_+}$, $\mathrm{Id}_{X_+} \wedge ev$, ϵ similarly. The map $\tilde{\Delta}_X^2 \times \mathrm{Id}_X$ extends to a map on the Thom spaces

$$\mathrm{Th}(\tilde{\Delta}_X^2 \times \mathrm{Id}_X) : \mathrm{Th}(p_2^{23*} \tilde{\nu}) \rightarrow \mathrm{Th}(p_2^{123*} \tilde{\nu}).$$

Let

$$\Delta_{13} : T^M \wedge X_+ \rightarrow X_+ \wedge T^M \wedge X_+$$

be the map induced from the map $T^M \times X_+ \rightarrow X_+ \times T^M \times X_+$ given by $(t, x) \mapsto (x, t, x)$.

We have the commutative diagram in $\mathcal{H}_\bullet(k)$

(1.11)

$$\begin{array}{ccccc} & & & \mathrm{Id}_{X_+} \wedge ev & \\ & & & \curvearrowright & \\ T^M \wedge X_+ & \xrightarrow{\delta \wedge \mathrm{Id}_{X_+}} & \mathrm{Th}(p_2^{123*} \tilde{\nu}) & \xrightarrow{\mathrm{Id}_{X_+} \wedge \epsilon} & X_+ \wedge T^M \wedge X_+ \\ & \searrow \eta \wedge \mathrm{Id}_{X_+} & \uparrow \mathrm{Th}(\tilde{\Delta}_X^2 \times \mathrm{Id}_X) & \uparrow \Delta_{13} & \searrow \mathrm{Id} \wedge \pi_X \\ & & \mathrm{Th}(p_2^{23*} \tilde{\nu}) & \xrightarrow{\epsilon} & T^M \wedge X_+ \xrightarrow{\tau_{T^M, X_+}} X_+ \wedge T^M \end{array}$$

Letting $f : X \rightarrow \mathrm{Spec} k$ be the structure morphism, the sequence

$$T^M \wedge X_+ \xrightarrow{\eta \wedge \mathrm{Id}_{X_+}} \mathrm{Th}(p_2^{23*} \tilde{\nu}) \xrightarrow{\epsilon} T^M \wedge X_+$$

is isomorphic to the sequence [18, (5.23)] applied to $\mathbb{S}_X \in \mathrm{SH}(X)$, when one makes the following identifications with the notation of [18]: $\Sigma^{\mathcal{M}}$ is Σ_T^M and we replace $\Sigma^{\mathcal{N}}$ with $p_{X\#} \circ (\mathrm{Th}_{\tilde{X}} \tilde{\nu}_{\tilde{X}} \wedge (-)) \circ p_X^*$. These two functors are canonically isomorphic via the canonical isomorphism [18, (5.18)]

$$p_X^* \mathrm{Th}_X(\mathcal{N}) \cong \mathrm{Th}_X(p_X^* \mathcal{N}) \cong \mathrm{Th}_{\tilde{X}}(\tilde{\nu}_{\tilde{X}}),$$

noting that $p_X : \tilde{X} \rightarrow X$ is an affine space bundle.

Via this identification, [18, Theorem 5.22] implies

$$\epsilon \circ (\eta \wedge \mathrm{Id}_{X_+}) = \mathrm{Id}_{T^M \wedge X_+},$$

which together with the commutativity of (1.11) yields the identity (1.10). \square

The triple $(\Sigma_T^\infty X_+^{\tilde{\vee}}, \delta_X, ev_X)$ is thus uniquely determined, up to unique isomorphism, by $\Sigma_T^\infty X_+$. In the projective case, we will denote $\Sigma_T^\infty X_+^{\tilde{\vee}}$ by $\Sigma_T^\infty X_+^\vee$ and will freely use the usual properties of the dual.

Definition 1.4. Let X be a smooth projective k -scheme. $\chi^{cat}(X)$ is defined to be the element $ev_X \circ \tau_{X, X^\vee} \circ \delta_X$ of $[\mathbb{S}_k, \mathbb{S}_k]$, where

$$\tau_{X, X^\vee} : \Sigma_T^\infty X_+ \wedge (\Sigma_T^\infty X_+)^\vee \rightarrow (\Sigma_T^\infty X_+)^\vee \wedge \Sigma_T^\infty X_+$$

is the symmetry isomorphism.

Via Morel's isomorphism $[\mathbb{S}_k, \mathbb{S}_k] \cong \mathrm{GW}(k)$ [25, Theorem 6.4.1], we consider $\chi^{cat}(X)$ as an element of $\mathrm{GW}(k)$.

We have the isomorphism

$$(1.12) \quad p_X^* T_X \xrightarrow{di_2} N_{\tilde{\Delta}_X^2}$$

defined similarly to the isomorphism (1.8), using the inclusion

$$T_X = \Delta_X^* p_2^* T_X \xrightarrow{di_2} \Delta_X^* (p_1^* T_X \oplus p_2^* T_X) \cong \Delta_X^* T_{X \times X} \xrightarrow{\rho} N_{\Delta_X}$$

instead of di_1 . As above, Combining (1.12) and (1.6) gives the isomorphism

$$(1.13) \quad N_{\tilde{\Delta}_X^2} \oplus \tilde{\nu}_{\tilde{X}} \xrightarrow{\alpha_2} i_{\tilde{X}}^* (p^* T_{\mathbb{P}^d} \oplus \tilde{\nu}_{\mathbb{P}^d}) \cong O_{\tilde{X}}^{d^2+2d},$$

The inclusion $\tilde{\nu}_{\tilde{X}} \rightarrow N_{\tilde{\Delta}_X^2} \oplus \tilde{\nu}_{\tilde{X}}$ and isomorphism (1.13) induce the inclusion of vector bundles over \tilde{X}

$$\beta_{X \subset \mathbb{P}^d} : \tilde{\nu}_{\tilde{X}} \rightarrow O_{\tilde{X}}^{d^2+2d}$$

and thereby the map on the Thom spaces

$$\mathrm{Th}(\beta_{X \subset \mathbb{P}^d}) : \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \rightarrow \Sigma_T^{d^2+2d} \tilde{X}_+.$$

Lemma 1.5. $\chi^{cat}(X)$ is equal to the stabilization and desuspension of the composition

$$T^{d^2+2d} \xrightarrow{\eta_{X \subset \mathbb{P}^d}} \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \xrightarrow{\mathrm{Th}(\beta_{X \subset \mathbb{P}^d})} \Sigma_T^{d^2+2d} \tilde{X}_+ \xrightarrow{\Sigma_T^{d^2+2d} \pi_{\tilde{X}}} T^{d^2+2d}$$

Proof. Let $s_0^2 : X \times \tilde{X} \rightarrow q_2^* \tilde{\nu}_{\tilde{X}}$ be the 0-section. Then $\text{Th}(\beta_{X \subset \mathbb{P}^d})$ is equal to the composition

$$\begin{aligned} \text{Th}(\tilde{\nu}_{\tilde{X}}) &\xrightarrow{\text{Th}(\tilde{\Delta}_X^2)} \text{Th}(q_2^*(\tilde{\nu}_{\tilde{X}})) \\ &\xrightarrow{\pi_2} \text{Th}(q_2^*(\tilde{\nu}_{\tilde{X}}))/[\text{Th}(q_2^*(\tilde{\nu}_{\tilde{X}})) \setminus s_0^2(\tilde{X})] \cong \text{Th}(\tilde{N}_{\tilde{\Delta}_X^2} \oplus \nu_{\tilde{X}}) \\ &\xrightarrow{\text{Th}(\alpha_2)} \text{Th}_{\tilde{X}}(p_X^* T_X \oplus \tilde{\nu}_{\tilde{X}}) \cong \Sigma_T^{d^2+2d} \tilde{X}_+ \end{aligned}$$

The symmetry isomorphism $\tau_{X, \tilde{X}} : X \times \tilde{X} \rightarrow \tilde{X} \times X$ induces the isomorphism on the Thom spaces

$$\text{Th}(\tau_{X, \tilde{X}}) : \text{Th}(q_2^* \tilde{\nu}_{\tilde{X}}) \rightarrow \text{Th}(q_1^* \tilde{\nu}_{\tilde{X}}),$$

which becomes the symmetry isomorphism τ_{X, X^\vee} upon stabilization. We have the diagram

$$\begin{array}{ccccc} T^{d^2+2d} & \xrightarrow{\delta_{X \subset \mathbb{P}^d}} & \text{Th}(q_2^* \tilde{\nu}_{\tilde{X}}) & \xrightarrow{\text{Th}(\tau_{X, \tilde{X}})} & \text{Th}(q_1^* \tilde{\nu}_{\tilde{X}}) \\ & \searrow \eta_{X \subset \mathbb{P}^d} & \uparrow \text{Th}(\tilde{\Delta}_X^2) & \text{Th}(\tau_{X, \tilde{X}}) \uparrow & \searrow \epsilon_{X \subset \mathbb{P}^d} \\ & & \text{Th}(\tilde{\nu}_{\tilde{X}}) & \xrightarrow{\text{Th}(\tilde{\Delta}_X^2)} & \text{Th}(q_2^* \tilde{\nu}_{\tilde{X}}) \xrightarrow{\text{Th}(\alpha_2) \circ \pi_2} \Sigma_T^{d^2+2d} X_+ \end{array}$$

which commutes since $\epsilon_{X \subset \mathbb{P}^d} = \text{Th}(\alpha_1) \circ \pi_1$. The result follows from this. \square

Remark 1.6 (A remark on the purity isomorphism). Suppose we have closed immersions $i_Y : Y \rightarrow U$, $i_X : X \rightarrow U$ in \mathbf{Sm}/k with $i_Y(Y) \subset X$, giving the closed immersion $i_{Y/X} : Y \rightarrow X$. Composing the zero-section $s_0^X : X \rightarrow N_{i_X}$ with $i_{Y/X}$ gives the closed immersion $s_0^{X|Y} : Y \rightarrow N_{i_X}$. Let $N_{Y:X}$ be the normal bundle of $s_0^{X|Y}(Y)$ in N_{i_X} . We have a canonical isomorphism

$$\psi : N_{Y:X} \rightarrow N_{i_Y},$$

purity isomorphisms

$$\begin{aligned} \theta_{i_X} &: U/U \setminus i_X(X) \rightarrow \text{Th}(N_{i_X}) \\ \theta_{i_Y} &: U/U \setminus i_Y(Y) \rightarrow \text{Th}(N_{i_Y}) \\ \theta_{i_{Y/X}} &: \text{Th}(N_{i_X})/\text{Th}(N_{i_X}) \setminus s_0^{X|Y}(Y) \rightarrow \text{Th}(N_{Y:X}) \end{aligned}$$

and the quotient maps

$$\begin{aligned} \pi_{Y/X} &: U/U \setminus i_X(X) \rightarrow U/U \setminus i_Y(Y) \\ \pi_{\text{Th}(Y/X)} &: \text{Th}(N_{i_X}) \rightarrow \text{Th}(N_{i_X})/\text{Th}(N_{i_X}) \setminus s_0^{X|Y}(Y). \end{aligned}$$

Then the diagram

$$(1.14) \quad \begin{array}{ccc} U/U \setminus i_X(X) & \xrightarrow{\theta_{i_X}} & \mathrm{Th}(N_{i_X}) \\ \pi_{Y/X} \downarrow & & \downarrow \mathrm{th}(\psi) \circ \theta_{i_{Y/X}} \circ \pi_{\mathrm{Th}(Y/X)} \\ U/U \setminus i_Y(Y) & \xrightarrow{\theta_{i_Y}} & \mathrm{Th}(N_{i_Y}) \end{array}$$

commutes.

To see this, we recall that for each closed immersion $i : Z \rightarrow W$ in \mathbf{Sm}/k one has the deformation diagram $Z \times \mathbb{A}^1 \xrightarrow{s_Z} \mathrm{Def}(i) \xrightarrow{p} W \times \mathbb{A}^1$, used by Morel-Voevodsky in their proof of the purity theorem [27, Theorem 3.2.23]. $\mathrm{Def}(i)$ is constructed by blowing up $W \times \mathbb{A}^1$ along $Z \times 0$ and then removing the proper transform of $W \times 0$, this defines the morphism $p : \mathrm{Def}(i) \rightarrow W \times \mathbb{A}^1$. The closed immersion $Z \times \mathbb{A}^1 \rightarrow W \times \mathbb{A}^1$ lifts uniquely to a closed immersion $s_Z : Z \times \mathbb{A}^1 \rightarrow \mathrm{Def}(i)$.

We take the blow-up $\widetilde{U \times \mathbb{A}^1 \times \mathbb{A}^1}$ of $U \times \mathbb{A}^1 \times \mathbb{A}^1$ along $X \times 0 \times \mathbb{A}^1$, then blow up $\widetilde{U \times \mathbb{A}^1 \times \mathbb{A}^1}$ along the proper transform of $Y \times \mathbb{A}^1 \times 0$, and finally remove the proper transforms of $U \times 0 \times \mathbb{A}^1$ and $U \times \mathbb{A}^1 \times 0$. The resulting scheme $\mathrm{Def}(i_X, i_Y)$ is a smooth scheme over $\mathbb{A}^1 \times \mathbb{A}^1$,

$$\pi : \mathrm{Def}(i_X, i_Y) \rightarrow \mathbb{A}^1 \times \mathbb{A}^1.$$

factoring the projection $p : \mathrm{Def}(i_X, i_Y) \rightarrow U \times \mathbb{A}^1 \times \mathbb{A}^1$. p admits a section \tilde{s}_Y over $Y \times \mathbb{A}^1 \times \mathbb{A}^1$ and \tilde{s}_X over $X \times \mathbb{A}^1 \times 1$.

The restriction $(p^{-1}(U \times \mathbb{A}^1 \times 1), \tilde{s}_X)$ is the deformation diagram $\mathrm{Def}(i_X)$ for $i_X : X \rightarrow U$, $(p^{-1}(U \times 1 \times \mathbb{A}^1), \tilde{s}_{Y|Y \times 1 \times \mathbb{A}^1})$ is the deformation diagram $\mathrm{Def}(i_Y)$ for $i_Y : Y \rightarrow U$ and $(p^{-1}(U \times 0 \times \mathbb{A}^1), \tilde{s}_{Y|Y \times 0 \times \mathbb{A}^1})$ is the deformation diagram $\mathrm{Def}(N_{i_X})$ for $N_{i_X} : Y \rightarrow N_{i_X}$. The Morel-Voevodsky purity theorem [27, Theorem 3.2.23] shows that the inclusion at $(1, 1)$ induces an isomorphism in $\mathcal{H}_\bullet(k)$

$$U/U \setminus Y \xrightarrow{i_1^Y} \mathrm{Def}(U, X, Y) / \mathrm{Def}(U, X, Y) \setminus s_Y(Y \times \mathbb{A}^1 \times \mathbb{A}^1)$$

the inclusion at $(0, 0)$ induces an isomorphism in $\mathcal{H}_\bullet(k)$

$$\mathrm{Th}(N_{Y:X}) \xrightarrow{i_0^N} \mathrm{Def}(U, X, Y) / \mathrm{Def}(U, X, Y) \setminus s_Y(Y \times \mathbb{A}^1 \times \mathbb{A}^1)$$

the inclusion at $(1, 0)$ induces an isomorphism in $\mathcal{H}_\bullet(k)$

$$\mathrm{Th}(N_{i_Y}) \xrightarrow{i_1^Y} \mathrm{Def}(U, X, Y) / \mathrm{Def}(U, X, Y) \setminus s_Y(Y \times \mathbb{A}^1 \times \mathbb{A}^1)$$

and the inclusion at $(0, 1)$ induces an isomorphism in $\mathcal{H}_\bullet(k)$

$$\begin{aligned} & \mathrm{Th}(N_{i_X}) / \mathrm{Th}(N_{i_X}) \setminus N_{i_X}(Y) \\ & \xrightarrow{i_1^N} \mathrm{Def}(U, X, Y) / \mathrm{Def}(U, X, Y) \setminus s_Y(Y \times \mathbb{A}^1 \times \mathbb{A}^1). \end{aligned}$$

Similarly, the inclusion $(1, 1)$ induces an isomorphism in $\mathcal{H}_\bullet(k)$

$$U/U \setminus X \xrightarrow{i_1^X} p^{-1}(U \times \mathbb{A}^1 \times 1)/p^{-1}(U \times \mathbb{A}^1 \times 1) \setminus s_X(X \times \mathbb{A}^1 \times 1)$$

and the inclusion at $(0, 1)$ induces an isomorphism in $\mathcal{H}_\bullet(k)$

$$\mathrm{Th}(N_{i_X}) \xrightarrow{i_0^X} p^{-1}(U \times \mathbb{A}^1 \times 1)/p^{-1}(U \times \mathbb{A}^1 \times 1) \setminus s_X(X \times \mathbb{A}^1).$$

The purity isomorphism θ_{i_X} is by definition $(i_0^X)^{-1} \circ i_1^X$. As the deformation diagram for i_Y is $(p^{-1}(U \times 1 \times \mathbb{A}^1), \tilde{s}_{Y|Y \times 1 \times \mathbb{A}^1})$, we have similarly $\theta_{i_Y} = (i_0^Y)^{-1} \circ i_1^Y$. For the same reason, the purity isomorphism $\theta_{i_{Y/X}}$ is $(i_0^N)^{-1} \circ i_1^N$. The quotient maps $\pi_{Y/X}$ and $\pi_{\mathrm{Th}(Y/X)}$ are induced by taking the inclusion $\mathrm{Def}(i_X) \subset \mathrm{Def}(i_X, i_Y)$ and replacing $s_X(X \times \mathbb{A}^1)$ with $s_Y(Y \times \mathbb{A}^1 \times \mathbb{A}^1)$. The commutativity of the diagram (1.14) follows from these facts.

1.2. Properties of the categorical Euler characteristic. Hoyois has initiated the systematic investigation of the categorical Euler characteristic in [17]. In this section we outline some of the properties of χ^{cat} that are essentially formal consequences of duality in a symmetric monoidal category.

Let $(\mathcal{C}, \wedge, \mathbf{1})$ be a symmetric monoidal category, and let $R_{\mathcal{C}} = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$. A *dual* of an object X of \mathcal{C} is a triple (Y, δ, ev) , $\delta : \mathbf{1} \rightarrow X \wedge Y$, $ev : Y \wedge X \rightarrow \mathbf{1}$ such that $(\mathrm{Id}_X \wedge ev) \circ (\delta \wedge \mathrm{Id}_X) = \mathrm{Id}_X$ and $(ev \wedge \mathrm{Id}_Y) \circ (\mathrm{Id}_Y \wedge \delta) = \mathrm{Id}_Y$. A dual of X , if it exists, is unique up to unique isomorphism. Assuming that the symmetric monoidal product \wedge admits a right adjoint $\mathcal{H}om$, one always has the weak dual $X^\vee := \mathcal{H}om(X, \mathbf{1})$, with a canonical map

$$ev_X : X \wedge X^\vee \rightarrow \mathbf{1}.$$

X will admit a dual (X^\vee, δ_X, ev_X) if and only if the canonical map $X^\vee \wedge X \rightarrow \mathcal{H}om(X, X)$ induced by ev_X is an isomorphism (see [23, Theorem 2.6]). Such an object is called *strongly dualizable*; to keep the terminology uniform, we will also call an object X in an arbitrary symmetric monoidal category strongly dualizable if X admits a dual (X^\vee, δ_X, ev_X) .

As above, if X is strongly dualizable, one has the Euler characteristic

$$\chi^{\mathrm{cat}}(X) := ev_X \circ \tau_{X, X^\vee} \circ \delta_X \in R_{\mathcal{C}}.$$

Clearly the collection of strongly dualizable objects in \mathcal{C} is closed under \wedge and

$$\begin{aligned} \chi^{\mathrm{cat}}(X \wedge Y) &= \chi^{\mathrm{cat}}(X) \cdot \chi^{\mathrm{cat}}(Y), \\ \chi^{\mathrm{cat}}(X^\vee) &= \chi^{\mathrm{cat}}(X), \end{aligned}$$

for strongly dualizable objects X and Y .

If \mathcal{C} is a triangulated tensor category, then the collection of strongly dualizable objects in \mathcal{C} forms the objects in a thick subcategory of \mathcal{C} , and the Euler characteristic is additive in distinguished triangles: if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle of strongly dualizable objects, then

$\chi^{cat}(B) = \chi^{cat}(A) + \chi^{cat}(C)$, and $\chi^{cat}(A[1]) = -\chi^{cat}(A)$. See [22, Theorem 0.1].

In the case $\mathcal{C} = \mathbf{SH}(k)$, the suspension spectrum $\Sigma_T^\infty X_+$ for X a smooth projective k -scheme is strongly dualizable, as we have seen. We call $U \in \mathbf{Sm}/k$ *dualizable* if $\Sigma_T^\infty U_+$ is strongly dualizable. If k admits resolution of singularities, then all $U \in \mathbf{Sm}/k$ are dualizable [30, Théorème 1.4]. Similarly, we call $\mathcal{X} \in \mathbf{Spc}(k)$ or $\mathcal{Y} \in \mathbf{Spc}_\bullet(k)$ dualizable if $\Sigma_T^\infty \mathcal{X}_+$ or $\Sigma_T^\infty \mathcal{Y}$ is strongly dualizable.

All the spheres $S^{p,q} \in \mathbf{Spc}_\bullet(k)$ are dualizable, in particular $\mathbb{A}^n \setminus \{0\}$ is dualizable for all n and $T = \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})$ is dualizable.

Lemma 1.7. $\chi^{cat}(S^{p,q}) = (-1)^p \cdot \langle -1 \rangle^q$.

Proof. As $T \cong S^{2,1}$, the formal properties of χ^{cat} mentioned above reduce this to showing $\chi^{cat}(T) = \langle -1 \rangle$. To simplify the notation, we write T^n for $\Sigma_T^n S_k$, for $n \in \mathbb{Z}$. Then $T^\vee = T^{-1}$ and the maps $\delta_T : 1 \rightarrow T \wedge T^{-1}$ and $\epsilon_T : T^{-1} \wedge T \rightarrow 1$ are the canonical isomorphisms. Thus $\chi^{cat}(T)$ is the composition

$$1 \cong T \wedge T^{-1} \xrightarrow{\tau_{T,T^{-1}}} T^{-1} \wedge T \cong 1$$

where the two isomorphisms are the canonical ones.

We claim that $\Sigma_T^2 \chi^{cat}(T)$ is the automorphism $\tau_{T,T}$ of T^2 . Morel [25, Remark 6.3.5] has shown $\tau_{T,T} = \times \langle -1 \rangle$, so the lemma follows from this claim.

Let $c_T : T^3 \rightarrow T^3$ be the automorphism corresponding to the cyclic permutation (123), let $c_4 : T^2 \wedge T^{-1} \wedge T \rightarrow T \wedge T^{-1} \wedge T^2$ be the symmetry isomorphism corresponding to the cyclic permutation (1432), and let $c_3 : T^2 \wedge 1 \rightarrow T \wedge 1 \wedge T$ be the symmetry isomorphism corresponding to the cyclic permutation (132). Voevodsky has remarked [36, Lemma 4.4] that $c_T = \text{Id}_{T^3}$ (see [20, Lemma 3.13] for a proof by Jardine). We thus have the commutative diagram

$$\begin{array}{ccccccc}
T^2 \wedge 1_{(1)} & \xrightarrow{\text{Id}_{T^2} \wedge \delta_T} & T^3 \wedge T^{-1} & \xrightarrow{\text{Id}_{T^2} \wedge \tau_{T,T^{-1}}} & T^2 \wedge T^{-1} \wedge T & \xrightarrow{\text{Id}_{T^2} \wedge \epsilon_T} & T^2 \wedge 1_{(2)} \\
& \searrow \text{Id}_{T^2} \wedge \delta_T & \downarrow c_T \wedge \text{Id} & & \downarrow c_4 & & \downarrow c_3 \\
& & T^3 \wedge T^{-1} & \xrightarrow{\tau_{T^2, T \wedge T^{-1}}} & T \wedge T^{-1} \wedge T^2 & \xrightarrow{\text{Id} \wedge \epsilon_T \wedge \text{Id}} & T \wedge 1 \wedge T \\
& & & & & & \nearrow \text{Id} \wedge \tau_{1,T} \\
& & & & & & T^2 \wedge 1_{(3)}
\end{array}$$

$\nwarrow \tau_{T,T} \wedge \text{Id}$

We add the subscripts (1), (2) and (3) so we can refer to these three copies of $T^2 \wedge 1$. The map $T^2 \wedge 1_{(1)} \rightarrow T^2 \wedge 1_{(3)}$ is the identity, as one sees by following along the lower part of the diagram, and thus the map $T^2 \wedge 1_{(1)} \rightarrow T^2 \wedge 1_{(2)}$ by going counter-clockwise around the outside of the diagram is $\tau_{T,T} \wedge \text{Id}_1$. The map $T^2 \wedge 1_{(1)} \rightarrow T^2 \wedge 1_{(2)}$ along the top is $\text{Id}_{T^2} \wedge \chi^{cat}(T)$, which proves the claim. \square

Remark 1.8. It is not difficult to see directly that $c_T : T^3 \rightarrow T^3$ is the identity in $\mathcal{H}_\bullet(k)$: one identifies T^3 with $\mathbb{A}^3/\mathbb{A}^3 \setminus \{0\}$ using the purity isomorphism,

via which c_T becomes the map induced by the linear transformation $\mathbb{A}^3 \rightarrow \mathbb{A}^3$ sending (x, y, z) to (z, x, y) . As this linear map is in $\mathrm{SL}_3(k)$ and SL_3 is \mathbb{A}^1 -connected, it follows that $c_T = \mathrm{Id}$ in $\mathcal{H}_\bullet(k)$. Similarly, the symmetry $\tau_{T,T} : T^2 \rightarrow T^2$ arises from the linear map $\mathbb{A}^2 \rightarrow \mathbb{A}^2$, $(x, y) \mapsto (y, x)$ via the purity isomorphism $T^2 \cong \mathbb{A}^2/\mathbb{A}^2 \setminus \{0\}$. Since $(x, y) \mapsto (y, -x)$ is in $\mathrm{SL}_2(k)$, and $\langle -1 \rangle$ is similarly induced by $x \mapsto -x$, the same argument shows that $\tau_{T,T} \circ (\langle -1 \rangle \wedge \mathrm{Id})$ is the identity on T^2 .

Definition 1.9. Let F , X and Y be in \mathbf{Sm}/k and let $p : Y \rightarrow X$ be a Zariski locally trivial fiber bundle with fiber F (and group \mathcal{G}). We say that p is dualizably locally trivial if F is dualizable and X admits a (finite) trivializing open cover $\mathcal{U} = \{U_i\}$ for p such that all intersections $U_{i_0} \cap \dots \cap U_{i_n}$ are dualizable for all $n \geq 0$.

Of course, if k admits resolution of singularities, each Zariski locally trivial fiber bundle $p : Y \rightarrow X$ with X and Y in \mathbf{Sm}/k is dualizably locally trivial.

For a dualizable space $\mathcal{X} \in \mathbf{Spc}(k)$, we write $\chi^{\mathrm{cat}}(\mathcal{X})$ for $\chi^{\mathrm{cat}}(\Sigma_T^\infty \mathcal{X}_+)$. For a dualizable space $\mathcal{X} \in \mathbf{Spc}_\bullet(k)$ we similarly write $\chi^{\mathrm{cat}}(\mathcal{X})$ for $\chi^{\mathrm{cat}}(\Sigma_T^\infty \mathcal{X})$.

Proposition 1.10. 1. Let F , X and Y be in \mathbf{Sm}/k and let $p : Y \rightarrow X$ be a Zariski locally trivial fiber bundle with fiber F and group \mathcal{G} . Suppose p is dualizably locally trivial. Then

$$\chi^{\mathrm{cat}}(Y) = \chi^{\mathrm{cat}}(X) \cdot \chi^{\mathrm{cat}}(F).$$

2. Let X be in \mathbf{Sm}/k and let $p : V \rightarrow X$ be a rank r vector bundle that is dualizably locally trivial. Then $\mathrm{Th}(V)$ is dualizable and

$$\chi^{\mathrm{cat}}(\mathrm{Th}(V)) = \langle -1 \rangle^r \chi^{\mathrm{cat}}(X).$$

3. Let X be in \mathbf{Sm}/k and let $p : V \rightarrow X$ be a rank r vector bundle that is dualizably locally trivial. Let $q : \mathbb{P}(V) \rightarrow X$ be the associated projective space bundle $\mathrm{Proj}_X(\mathrm{Sym}^* V^\vee)$. Then $\mathbb{P}(V)$ is dualizable and

$$\chi^{\mathrm{cat}}(\mathbb{P}(V)) = \left(\sum_{i=0}^{r-1} \langle -1 \rangle^i \right) \chi^{\mathrm{cat}}(X).$$

4. Let $i : Z \rightarrow X$ be a codimension c closed immersion in \mathbf{Sm}/k . Let $p : N_Z \rightarrow Z$ be the normal bundle of i and suppose that N_Z is dualizably locally trivial as a vector bundle over Z . Suppose in addition that X is dualizable. Let \tilde{X} be the blow up of X along Z . Then \tilde{X} is dualizable and

$$\chi^{\mathrm{cat}}(\tilde{X}) = \chi^{\mathrm{cat}}(X) + \left(\sum_{i=1}^{c-1} \langle -1 \rangle^i \right) \chi^{\mathrm{cat}}(Z).$$

5. Let $\sigma : k \rightarrow F$ be an extension of fields, inducing the homomorphism $\sigma_* : \mathrm{GW}(k) \rightarrow \mathrm{GW}(F)$. Then for a dualizable $X \in \mathbf{Sm}/k$, the base-extension $X_F \in \mathbf{Sm}/F$ is dualizable and

$$\chi^{\mathrm{cat}}(X_F) = \sigma_*(\chi^{\mathrm{cat}}(X)).$$

Proof. We first note that, if X has a finite Zariski open cover $\mathcal{U} = \{U_i\}$ such that $U_{i_0} \cap \dots \cap U_{i_n}$ is dualizable for all $n \geq 0$, then X is itself dualizable: this follows easily from the Mayer-Vietoris distinguished triangle and induction on the number of elements in \mathcal{U} . Similarly, if $U_{i_0} \cap \dots \cap U_{i_n}$ is dualizable and F is dualizable, then $(U_{i_0} \cap \dots \cap U_{i_n}) \times F$ is also dualizable and thus Y is dualizable. Since

$$\chi^{cat}((U_{i_0} \cap \dots \cap U_{i_n}) \times F) = \chi^{cat}(U_{i_0} \cap \dots \cap U_{i_n}) \cdot \chi^{cat}(F)$$

the additivity of χ^{cat} in distinguished triangles together with the Mayer-Vietoris triangles for \mathcal{U} and for $\mathcal{V} := \{U_i \times F\}$ shows that

$$\chi^{cat}(Y) = \chi^{cat}(X) \cdot \chi^{cat}(F).$$

For (2), the distinguished triangle

$$\Sigma_T^\infty(V \setminus 0_X)_+ \rightarrow \Sigma_T^\infty V_+ \rightarrow \Sigma_T^\infty \mathrm{Th}(V) \rightarrow$$

gives

$$\chi^{cat}(\mathrm{Th}(V)) = \chi^{cat}(X) - \chi^{cat}(V \setminus 0_X)$$

Since $p : V \rightarrow X$ is dualizably locally trivial, so is $V \setminus 0_X \rightarrow X$, so

$$\chi^{cat}(V \setminus 0_X) = \chi^{cat}(\mathbb{A}^r \times X \setminus 0_X)$$

and thus

$$\chi^{cat}(\mathrm{Th}(V)) = \chi^{cat}(\mathrm{Th}(O_X^r)) = \chi^{cat}(T^{\wedge r} \wedge X_+) = \chi^{cat}(T)^r \cdot \chi^{cat}(X).$$

By Lemma 1.7, $\chi^{cat}(T) = \langle -1 \rangle$ and thus

$$\chi^{cat}(\mathrm{Th}(V)) = \langle -1 \rangle^r \chi^{cat}(X).$$

For (3), we again reduce to the case $\mathbb{P}(V) = \mathbb{P}^{r-1} \times X$, which reduces to the computation of $\chi^{cat}(\mathbb{P}^{r-1})$. We proceed by induction in r , using the distinguished triangle

$$\Sigma_T^\infty \mathbb{P}_+^{r-2} \rightarrow \Sigma_T^\infty \mathbb{P}_+^{r-1} \rightarrow T^{\wedge r-1} \wedge \Sigma_T^\infty \mathrm{Spec} k_+ \rightarrow$$

and (2).

For (4), the assumption that $N_Z \rightarrow Z$ is dualizably locally trivial implies by homotopy invariance that Z is dualizable and that $\mathrm{Th}(N_Z)$ is dualizable. Let $E \subset \tilde{X}$ be the exceptional divisor, so $E = \mathbb{P}(N_Z)$ and thus E is dualizable. Let $N_E \rightarrow E$ be the normal of E in \tilde{X} , and let $q : E \rightarrow Z$ be the projection. If $\mathcal{U} = \{U_i\}$ is a trivializing open cover of Z for $p : N_Z \rightarrow Z$, then $q^{-1}(U_i) \cong U_i \times \mathbb{P}^{c-1}$. Via this isomorphism, we have $N_{E|q^{-1}(U_i)} \cong p_2^* \mathcal{O}_{\mathbb{P}^{c-1}}(1)$, so this bundle is locally trivialized on $q^{-1}(U_i)$ by the pullback of the standard affine open cover of \mathbb{P}^{c-1} . Thus for all indices i_0, \dots, i_n , $N_{E|q^{-1}(\cap_j U_{i_j})}$ and $(N_E \setminus 0_E)_{|q^{-1}(\cap_j U_{i_j})}$ are dualizably locally trivial. Thus $\mathrm{Th}(N_E)$ is dualizable and

$$\chi^{cat}(\mathrm{Th}(N_E)) = \langle -1 \rangle \chi^{cat}(E), \quad \chi^{cat}(\mathrm{Th}(N_Z)) = \langle -1 \rangle^c \chi^{cat}(Z)$$

by (2).

We have the purity isomorphisms

$$X/U \cong \mathrm{Th}(N_Z), \tilde{X}/U \cong \mathrm{Th}(N_E)$$

giving the distinguished triangles

$$\begin{aligned} \Sigma_T^\infty U_+ &\rightarrow \Sigma_T^\infty X_+ \rightarrow \mathrm{Th}(N_X) \rightarrow \\ \Sigma_T^\infty U_+ &\rightarrow \Sigma_T^\infty \tilde{X}_+ \rightarrow \mathrm{Th}(N_E) \rightarrow . \end{aligned}$$

The first of these show that U is dualizable and then the second one shows that \tilde{X} is dualizable.

Together with (2) and (3), these distinguished triangles yield the identities

$$\begin{aligned} \chi^{\mathrm{cat}}(\tilde{X}) - \chi^{\mathrm{cat}}(X) &= \chi^{\mathrm{cat}}(\mathrm{Th}(N_E)) - \chi^{\mathrm{cat}}(\mathrm{Th}(N_Z)) \\ &= \langle -1 \rangle \chi^{\mathrm{cat}}(\mathbb{P}(N_Z)) - \langle -1 \rangle^c \chi^{\mathrm{cat}}(Z) \\ &= \left(\sum_{i=1}^{c-1} \langle -1 \rangle^i \right) \chi^{\mathrm{cat}}(Z). \end{aligned}$$

For (5), let $\pi : \mathrm{Spec} F \rightarrow \mathrm{Spec} k$ be the morphism induced by σ . Then we have the symmetric monoidal exact functor $\pi^* : \mathrm{SH}(k) \rightarrow \mathrm{SH}(F)$, with $\pi^* \Sigma_T^\infty X_+ = \Sigma_T^\infty X_{F+}$ and with $\pi^* : \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) \rightarrow \mathrm{End}_{\mathrm{SH}(F)}(\mathbb{S}_F)$ the map $\sigma_* : \mathrm{GW}(k) \rightarrow \mathrm{GW}(F)$, via Morel's identification $\mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) \cong \mathrm{GW}(k)$, $\mathrm{End}_{\mathrm{SH}(F)}(\mathbb{S}_F) \cong \mathrm{GW}(F)$. Since π^* is compatible with duality, these facts prove (5). \square

Remarks 1.11. 1. The categorical Euler characteristic in an arbitrary symmetric monoidal category is clearly functorial with respect to symmetric monoidal functors. In particular, if $k = \mathbb{C}$, the image of $\chi^{\mathrm{cat}}(\mathcal{X})$ for a dualizable space $\mathcal{X} \in \mathbf{Spc}(\mathbb{C})$ under the Betti realization functor $Re_B : \mathrm{SH}(\mathbb{C}) \rightarrow \mathrm{SH}$ is the Euler characteristic of $Re_B(\mathcal{X})$ computed in SH . As the map $\mathrm{End}_{\mathrm{SH}}(\mathbb{S}) \rightarrow \mathrm{End}_{D(\mathbf{Ab})}(\mathbb{Z}) = \mathbb{Z}$ under the \mathbb{Z} -linearization map is an isomorphism, the Euler characteristic in SH of $\Sigma^\infty T_+$, for a finite CW complex T , is just the topological Euler characteristic of T . Since $\mathrm{GW}(\mathbb{C}) = \mathbb{Z}$ by rank, we see that, for $k \subset \mathbb{C}$, and for $X \in \mathbf{Sm}/k$, $\mathrm{rank} \chi^{\mathrm{cat}}(X)$ is the topological Euler characteristic of the complex manifold $X(\mathbb{C})^{\mathrm{an}}$ of \mathbb{C} -points of X .

We have as well the \mathbb{R} -Betti realization functor $Re_{B\mathbb{R}} : \mathrm{SH}(\mathbb{R}) \rightarrow \mathrm{SH}$, which for $X \in \mathbf{Sm}/\mathbb{R}$ sends the suspension spectrum $\Sigma_T^\infty X_+$ to the suspension spectrum of the real points of X , $\Sigma^\infty X(\mathbb{R})^{\mathrm{an}}$. We note that the induced map $\mathrm{GW}(\mathbb{R}) \rightarrow \mathrm{End}_{\mathrm{SH}}(\mathbb{S}) = \mathbb{Z}$ is the signature homomorphism. Indeed, we need only check that $\langle -1 \rangle$ goes to -1 . To see this, the map $\mathrm{GW}(k) \rightarrow \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$ is constructed by sending the one-dimensional form $\langle u \rangle$ to the automorphism m_u of \mathbb{P}^1 given by $[x_0 : x_1] \mapsto [x_0 : ux_1]$. On $\mathbb{P}^1(\mathbb{R})^{\mathrm{an}} = S^1$, m_{-1} is the map $\theta \mapsto -\theta$ and hence has degree -1 .² Concretely, for $X \in \mathbf{Sm}/\mathbb{R}$, the rank of $\chi^{\mathrm{cat}}(X)$ is the Euler characteristic of $X(\mathbb{C})^{\mathrm{an}}$ and the signature of $\chi^{\mathrm{cat}}(X)$ is the Euler characteristic of $X(\mathbb{R})^{\mathrm{an}}$.

²I am grateful to Fabien Morel for this argument.

2. For $q \in \mathrm{GW}(\mathbb{R})$ with signature $\mathrm{sgn}(q)$, one has $\mathrm{rank}(q) \equiv \mathrm{sgn}(q) \pmod{2}$. This implies that for $X \in \mathbf{Sm}/\mathbb{R}$, the Euler characteristic of $X(\mathbb{C})^{\mathrm{an}}$ and $X(\mathbb{R})^{\mathrm{an}}$ are congruent modulo 2. At least for proper \mathbb{R} -schemes, this is an easy consequence of the fact (see for example [24, pg. 76]) that for a compact Riemannian manifold M with an isometry $f : M \rightarrow M$, the fixed point locus M^f has Euler characteristic given by the Lefschetz number

$$\chi^{\mathrm{top}}(M^f) = \sum_i (-1)^i \mathrm{Tr}(f^*_{|H^i(M, \mathbb{Q})}).$$

One applies this to complex conjugation $\mathfrak{c} : X(\mathbb{C})^{\mathrm{an}} \rightarrow X(\mathbb{C})^{\mathrm{an}}$, after decomposing $H^i(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Q})$ into plus and minus eigenspaces for the action of \mathfrak{c} , to give the congruence. Probably this argument can be extended without much trouble to the case of open smooth varieties.

There is also an upper bound for $\chi^{\mathrm{top}}(X(\mathbb{R}))$ in terms of the Hodge theory of X , due to Abelson [1], namely, if X/\mathbb{R} is smooth and projective and has even dimension $2n$ over \mathbb{R} , then

$$|\chi^{\mathrm{top}}(X(\mathbb{R})^{\mathrm{an}})| \leq \dim_{\mathbb{C}} H^{n,n}(X_{\mathbb{C}}).$$

The proof uses the Hodge decomposition and the hard Lefschetz theorem. If $H^p(X(\mathbb{C})^{\mathrm{an}}, \mathbb{C}) = 0$ for all odd p , this gives the inequality $|\chi^{\mathrm{top}}(X(\mathbb{R})^{\mathrm{an}})| \leq \chi^{\mathrm{top}}(X(\mathbb{C})^{\mathrm{an}})$, in other words, that

$$|\mathrm{sgn}(\chi^{\mathrm{cat}}(X))| \leq \mathrm{rank}(\chi^{\mathrm{cat}}(X)).$$

On the other hand, this last inequality would follow if we knew that $\chi^{\mathrm{cat}}(X)$ were *effective*, that is, represented in $\mathrm{GW}(\mathbb{R})$ by a quadratic form rather than a difference of two quadratic forms. This raises the question: for which X , smooth and projective over a field k , is $\chi^{\mathrm{cat}}(k) \in \mathrm{GW}(k)$ effective? As all odd cohomology vanishes if for instance X is an even dimensional smooth complete intersection in a projective space, such varieties would be natural candidates for this property. See Example 11.13 for examples of varieties with effective Euler characteristic.

The following result is a simple consequence of the formal properties of the categorical Euler characteristic.

Proposition 1.12. *Let X be a smooth projective k -scheme of odd dimension over k . Then there is an integer m and a 2-torsion element $\tau \in \mathrm{GW}(k)$ such that*

$$\chi^{\mathrm{cat}}(X) = m \cdot h + \tau.$$

Proof. As $\chi^{\mathrm{cat}}(X)$ is additive with respect to disjoint union, we may assume that X is integral, and hence equi-dimensional over k . We first show that $\chi^{\mathrm{cat}}(X) \in \mathrm{GW}(k)$ has even rank. For this, we may replace k with its algebraic closure \bar{k} , and assume that k is algebraically closed, in which case $\mathrm{GW}(k) = \mathbb{Z}$ by rank. In addition, the unit map $\mathbb{S}_k \rightarrow H_{\mathrm{mot}}\mathbb{Z}$ induces the rank isomorphism $\mathrm{GW}(k) = \pi_{0,0}(\mathbb{S}_k) \cong \pi_{0,0}(H_{\mathrm{mot}}\mathbb{Z}) \cong H^0(\mathrm{Spec} k, \mathbb{Z}(0)) =$

\mathbb{Z} . Thus, it suffices to show that the categorical Euler characteristic of the motive $M(X) \in DM_{gm}(k)$ is an even integer. Choosing a prime ℓ prime to the characteristic of k , we may apply the ℓ -adic realization functor (with \mathbb{Q}_ℓ -coefficients) and as the induced map

$$\mathbb{Z} = H^0(\mathrm{Spec} k, \mathbb{Z}(0)) \rightarrow H_{\mathrm{\acute{e}t}}^0(\mathrm{Spec} k, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$$

is injective, it suffices to check that the ℓ -adic Euler characteristic of X is an even integer.

If X has dimension d , this reduces to showing that $\dim_{\mathbb{Q}_\ell} H^d(X, \mathbb{Q}_\ell)$ is an even integer. The cup product pairing

$$H^d(X, \mathbb{Q}_\ell) \times H^d(X, \mathbb{Q}_\ell) \rightarrow H^{2d}(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$$

is a perfect pairing, by Poincaré duality, and is alternating since d is odd, and thus $H^d(X, \mathbb{Q}_\ell)$ has even dimension, as claimed.

Let then $2m = \mathrm{rank} \chi^{\mathrm{cat}}(X)$, and consider the rank zero element $\chi^{\mathrm{cat}}(X) - m \cdot h$ of $\mathrm{GW}(k)$. We claim that

$$\langle -1 \rangle \cdot (\chi^{\mathrm{cat}}(X) - m \cdot h) = \chi^{\mathrm{cat}}(X) - m \cdot h.$$

Indeed, $\langle u \rangle \cdot h = h$ for any unit $u \in k^\times$, so this is equivalent to

$$(1.15) \quad \langle -1 \rangle \cdot \chi^{\mathrm{cat}}(X) = \chi^{\mathrm{cat}}(X).$$

But in a symmetric monoidal category \mathcal{C} , a strongly dualizable object \mathcal{X} satisfies $\chi^{\mathrm{cat}}(\mathcal{X}) = \chi^{\mathrm{cat}}(\mathcal{X}^\vee)$, and thus

$$\begin{aligned} \chi^{\mathrm{cat}}(X) &:= \chi^{\mathrm{cat}}(\Sigma_T^\infty X_+) \\ &= \chi^{\mathrm{cat}}((\Sigma_T^\infty X_+)^\vee) \\ &= \chi^{\mathrm{cat}}(\Sigma_T^\infty (\mathrm{Th}(-T_X))) \end{aligned}$$

where we write $\Sigma_T^\infty (\mathrm{Th}(-T_X))$ for $\Sigma_T^{-d^2-2d} \Sigma_T^\infty (\mathrm{Th}_{\tilde{X}}(\tilde{\nu}_{\tilde{X}}))$, after having taken a closed immersion $X \subset \mathbb{P}^d$. Note that $\tilde{\nu}_{\tilde{X}}$ has rank $d^2 + 2d - \dim_k X$, and it is not hard to see from the construction of $\tilde{\nu}_{\tilde{X}}$ that this bundle is always dualizably locally trivial on \tilde{X} . Thus

$$\chi^{\mathrm{cat}}(\Sigma_T^\infty (\mathrm{Th}(-T_X))) = (\langle -1 \rangle)^{\dim_k X} \chi^{\mathrm{cat}}(\Sigma_T^\infty X_+),$$

which together with the above identity yields the identity (1.15).

Noting that $\langle 1 \rangle - \langle -1 \rangle = 2 - h$ and $h \cdot q = \mathrm{rank}(q) \cdot h$ for all $q \in \mathrm{GW}(k)$, we find that

$$\begin{aligned} 0 &= (2 - h) \cdot (\chi^{\mathrm{cat}}(X) - m \cdot h) \\ &= 2 \cdot (\chi^{\mathrm{cat}}(X) - m \cdot h) \end{aligned}$$

and thus $\chi^{\mathrm{cat}}(X) - m \cdot h$ is a 2-torsion element of $\mathrm{GW}(k)$. \square

Remark 1.13. We will improve this result in Theorem 7.1, which shows that the 2-torsion element τ is actually zero.

One also has a simple expression for the Euler characteristic of a smooth cellular scheme. Recall that an integral finite type k -scheme X is *cellular* if X admits a filtration

$$\emptyset = X_{-1} \subset X_0 \subset \dots \subset X_n = X$$

with $X_i \setminus X_{i-1}$ a disjoint union of affine spaces \mathbb{A}_k^i . X_i is called the i -skeleton of the filtration.

We recall the following result of Hoyois' (private communication).

Proposition 1.14. *Let X be a smooth cellular k -scheme of dimension n with i skeleton X_i . Suppose that $X_i \setminus X_{i-1}$ is the disjoint union of m_i copies of \mathbb{A}^i . Then X is dualizable and*

$$\chi^{cat}(X) = \sum_{i=0}^n m_i \langle -1 \rangle^{n-i}.$$

Proof. Let d be the minimum i such that $X_{i+1} \neq \emptyset$; the proof is by downward induction on d . If $d = n$, then $X = \coprod^{m_n} \mathbb{A}^n$, which is isomorphic in $\mathcal{H}(k)$ to $\coprod^{m_n} \text{Spec } k$, proving the assertion in this case. If $d < n$, apply the induction hypothesis to $U := X \setminus X_d$. Then U is dualizable and

$$\chi^{cat}(U) = \sum_{i=d+1}^n m_i \langle -1 \rangle^{n-i}.$$

By the Quillen-Suslin theorem [29, 35], the normal bundle of X_d in X is $\mathcal{O}_{X_d(X)}^{n-d}$. Homotopy purity gives us the distinguished triangle

$$\Sigma_T^\infty U_+ \rightarrow \Sigma_T^\infty X_+ \rightarrow \Sigma_T^{n-d} \mathbb{S}_k^{\oplus m_d} \rightarrow \Sigma_T^\infty U_+[1],$$

which shows that X is dualizable; the formula for $\chi^{cat}(X)$ follows by the additivity of χ^{cat} in distinguished triangles [22, Theorem 0.1] and Lemma 1.7. \square

Examples 1.15. 1. As a simple example, Proposition 1.14 gives another proof that

$$\chi^{cat}(\mathbb{P}_k^n) = \begin{cases} \frac{n+1}{2} \cdot h & \text{for } n \text{ odd,} \\ \langle 1 \rangle + \frac{n}{2} \cdot h & \text{for } n \text{ even.} \end{cases}$$

This recovers Hoyois' computation of $\chi^{cat}(\mathbb{P}_k^n)$ [17, Example 1.7].

2. Let X be a Severi-Brauer variety over k of dimension n . The Euler characteristic of Severi-Brauer varieties have been computed by Hoyois (private communication). Using his quadratic refinement of the Lefschetz trace formula [17, Theorem 1.3] and the fact that for a central simple algebra \mathcal{A} over k , $\text{SL}_1(\mathcal{A})$ is \mathbb{A}^1 -connected, he shows

$$\chi^{cat}(X) = \chi^{cat}(\mathbb{P}_k^n).$$

In fact, the case of even n follows from the fact that X is split by a separable field extension $k \subset F$ of odd degree and $\text{GW}(k) \rightarrow \text{GW}(F)$ is injective if $[F : k]$ is odd and F/k is separable.

If X has odd dimension n , then by Proposition 1.12

$$\chi^{cat}(X) = \frac{n+1}{2} \cdot h + \tau = \chi^{cat}(\mathbb{P}_k^n) + \tau$$

for some 2-torsion element $\tau \in \mathrm{GW}(k)$. Our Theorem 7.1 improves Proposition 1.12, showing that $\tau = 0$.

2. EULER CLASSES AND THE CHOW-WITT EULER CHARACTERISTIC

2.1. The Grothendieck-Witt ring and Milnor-Witt sheaves. For a scheme U and a line bundle L on U , we have the Grothendieck-Witt group $\mathrm{GW}(U; L)$, defined by Balmer and Walter [10, 38] as the Witt group of the triangulated category of perfect complexes $D^{perf}(U)$, with duality $\mathcal{C}^\vee := \mathrm{Hom}(\mathcal{C}, L)$ and isomorphism $can : \mathcal{C} \rightarrow (\mathcal{C}^\vee)^\vee$ the canonical one; . The Grothendieck-Witt ring $\mathrm{GW}(U)$ is $\mathrm{GW}(U, \mathcal{O}_U)$, where the ring structure is induced by the tensor structure on $D^{perf}(U)$.

We recall (see [25, § 6.3]) that for a field F , the ring $K_*^{MW}(F)$ is the \mathbb{Z} -graded associative algebra with generators $[u]$, $u \in F^\times$, of degree one, and an additional generator η of degree -1 , with the following relations

- (1) $\eta[u] = [u]\eta$ for all $u \in F^\times$.
- (2) $[u][1-u] = 0$ for $u \in F \setminus \{0, 1\}$.
- (3) $[uv] = [u] + [v] + \eta[u][v]$ for all $u, v \in F^\times$.
- (4) $\eta(2 + \eta[-1]) = 0$

For $u \in F^\times$ we denote $1 + \eta[u]$ by $\langle u \rangle$, let $\epsilon = -\langle -1 \rangle$ and denote $2 + \eta[-1] = \langle 1 \rangle + \langle -1 \rangle$ by h . The relation (3) can be written as

$$(3') \quad [uv] = [u] + \langle u \rangle[v] = [v] + [u]\langle v \rangle$$

Since $[uv] = [vu]$, we have $[v] + [u]\langle v \rangle = [v] + \langle v \rangle[u]$, so $\langle v \rangle$ is in the center of $K_*^{MW}(F)$. Further relations (these may be found in [26, Lemma 2.7]) are:

- (5) $\langle uv \rangle = \langle u \rangle \langle v \rangle$.
- (6) $[u][-u] = 0$, $[u][u] = [u][-1]$.
- (7) $\langle uv^2 \rangle = \langle u \rangle$.
- (8) $[u^{-1}] = -\langle u^{-1} \rangle[u] = -\langle u \rangle[u] = -\langle -1 \rangle[u]$.
- (9) $[u][v] = \epsilon[v][u]$.
- (10) $\langle u \rangle h = h$.

In particular, sending $u \in F^\times$ to $\langle u \rangle$ defines a group homomorphism

$$\langle - \rangle : F^\times / (F^\times)^2 \rightarrow K_0^{MW}(F)^\times$$

Morel shows there is a unique isomorphism of rings,

$$(2.1) \quad \sigma_F : \mathrm{GW}(F) \rightarrow K_0^{MW}(F),$$

sending the one-dimensional form $q_u(x) = ux^2$ to $\langle u \rangle$. We will often write $\langle u \rangle$ for $q_u \in \mathrm{GW}(F)$ or more generally, for $q_u \in \mathrm{GW}(X)$ if u is a global unit on scheme X .

Morel [26, Part I, §2.2] has extended the definition of the Milnor-Witt ring of a field to a sheaf of graded rings on $\mathbf{Sm}/k_{\mathrm{Nis}}$, \mathcal{K}_*^{MW} . For $X \in \mathbf{Sm}/k$, we denote the restriction of \mathcal{K}_*^{MW} to X_{Nis} by \mathcal{K}_{X*}^{MW} (or simply \mathcal{K}_*^{MW} if the

context makes the meaning clear); it follows from the definition of \mathcal{K}_n^{MW} that for $x \in X \in \mathbf{Sm}/k$, X integral, the restriction map $\mathcal{K}_n^{MW}(\mathcal{O}_{X,x}) \rightarrow \mathcal{K}_n^{MW}(k(X))$ is injective. In addition, for $u \in \mathcal{O}_{X,x}^\times$, the element $[u] \in K_1^{MW}(k(X))$ lifts to $[u] \in \mathcal{K}_1^{MW}(\mathcal{O}_{X,x})$ and the relations (1)-(7) hold in $\mathcal{K}_*^{MW}(\mathcal{O}_{X,x})$ for units, with (2) requiring that u and $1 - u$ are units.

Let $\mathbb{Z}[\mathbb{G}_m]$ be the sheafification of the presheaf of rings on $\mathbf{Sm}/k_{\text{Nis}}$, $U \mapsto \mathbb{Z}[\mathcal{O}_U(U)^\times]$. For $L \rightarrow X$ a line bundle, we have the sheaf L^\times of nowhere zero sections of L , giving us the sheaf $\mathbb{Z}[L^\times]$ of $\mathbb{Z}[\mathbb{G}_m]_X$ -modules on X_{Nis} . Letting $u \in \mathcal{O}_{X,x}^\times$ act on $\mathcal{K}_n^{MW}(\mathcal{O}_{X,x})$ by $\times \langle u \rangle$ makes \mathcal{K}_{X*}^{MW} a sheaf of $\mathbb{Z}[\mathbb{G}_m]_X$ -modules; one defines

$$\mathcal{K}_*^{MW}(L)_X := \mathcal{K}_{X*}^{MW} \otimes_{\mathbb{Z}[\mathbb{G}_m]} \mathbb{Z}[L^\times].$$

Defining \mathcal{GW} to be the sheafification of the presheaf $U \mapsto \text{GW}(U)$ on $\mathbf{Sm}/k_{\text{Nis}}$, Morel's isomorphism (2.1) extends to an isomorphism $\mathcal{GW} \rightarrow \mathcal{K}_0^{MW}$. For L a line bundle on $X \in \mathbf{Sm}/k$, one has the sheafification $\mathcal{GW}_X(L)$ of the presheaf $U \mapsto \text{GW}(U; L)$ on X with $\mathcal{GW}_X(L) \cong \mathcal{GW}_X \otimes_{\mathbb{Z}[\mathbb{G}_m]} \mathbb{Z}[L^\times]$, giving the isomorphism $\mathcal{GW}_X(L) \cong \mathcal{K}_0^{MW}(L)_X$.

The sheaf $\mathcal{K}_n^{MW}(L)_X$ admits a Gersten resolution [5, Theorem 4.1.3]

$$\begin{aligned} (2.2) \quad 0 \rightarrow \mathcal{K}_n^{MW}(L)_X &\rightarrow \bigoplus_{x \in X^{(0)}} i_{x*} K_r^{MW}(L)(x) \rightarrow \dots \\ &\rightarrow \bigoplus_{x \in X^{(p)}} i_{x*} K_{n-p}^{MW}(L \otimes \det^{-1}(\mathfrak{m}_x/\mathfrak{m}_x^2))(k(x)) \rightarrow \dots \\ &\rightarrow \bigoplus_{x \in X^{(n)}} i_{x*} K_0^{MW}(L \otimes \det^{-1}(\mathfrak{m}_x/\mathfrak{m}_x^2))(k(x)) \rightarrow \dots, \end{aligned}$$

where as usual, for $x \in X$, $i_x : x \rightarrow X$ is the inclusion. We denote this by

$$0 \rightarrow \mathcal{K}_n^{MW}(L)_X \xrightarrow{\epsilon} \mathcal{C}^*(X, n, L).$$

Remark 2.1. Barge-Morel [11, §1] define a complex $C^*(X, J^n, L)$, studied further by Fasel, who denotes this complex by $C^*(X, G_n, L)$ [12, Définition 10.2.10]; in [4, Proposition 2.3.1] and [15, Theorem 5.4] it is shown that $C^*(X, G_n, L)$ is naturally isomorphic to the Gersten complex of global sections on X of $\mathcal{C}^*(X, n, L)$.

2.2. Twisted Milnor-Witt cohomology and Euler classes. We now turn to the definition of the Euler class of X via the Euler class of the tangent bundle.

For a morphism $f : Z \rightarrow Y$ in \mathbf{Sm}/k , one has the functorial pull-back map $f^* : \mathcal{K}_n^{MW}(L)_Y \rightarrow f_* \mathcal{K}_n^{MW}(f^* L)_Z$, inducing maps on (Nisnevich) cohomology

$$f^* : H^m(Y, \mathcal{K}_n^{MW}(L)) \rightarrow H^m(Z, \mathcal{K}_n^{MW}(f^* L)).$$

For $f : Z \rightarrow Y$ projective, of relative dimension d , Fasel [12, §12] has defined functorial pushforward maps

$$f_* : H^m(Z, \mathcal{K}_n^{MW}(f^* L \otimes \omega_{Z/k})) \rightarrow H^{m-d}(Y, \mathcal{K}_{n-d}^{MW}(L \otimes \omega_{Y/k})).$$

More precisely, Fasel defines the pushforward map as the map on cohomology induced by a map of complexes

$$f_* : C^*(Z, G_n, f^*L \otimes \omega_{Z/k}) \rightarrow C^*(X, G_n, L \otimes \omega_{Y/k})$$

which gives the above map on Milnor-Witt cohomology by the identification described in Remark 2.1.

For $\psi : L' \rightarrow L$ an isomorphism of line bundles on Y , the cartesian diagram

$$\begin{array}{ccc} L' & \xrightarrow{\psi} & L \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\text{Id}_Y} & Y \end{array}$$

identifies L' with $\text{Id}_Y^*(L)$, so we have the map

$$\psi^* := (\text{Id}_Y, \psi)^* : \mathcal{K}_n^{MW}(L)_Y \rightarrow \mathcal{K}_n^{MW}(L')_Y.$$

Let $\pi : V \rightarrow Y$ be a rank n vector bundle with zero-section $s_0 : Y \rightarrow V$. We have the determinant line bundle $\det V$ and its inverse $\det^{-1} V := (\det V)^{-1}$. The push-forward map

$$s_{0*} : H^m(Y, \mathcal{K}_p^{MW}) \rightarrow H^{m+n}(V, \mathcal{K}_{p+n}^{MW}(\pi^* \det^{-1} V))$$

is defined: we have the canonical isomorphism $\omega_{V/k} \cong \pi^*(\omega_{Y/k} \otimes \det^{-1}(V))$, so we have

$$\begin{aligned} H^m(Y, \mathcal{K}_p^{MW}) &= H^m(Y, \mathcal{K}_p^{MW}(s_0^* \pi^*(\omega_{Y/k}^{-1}) \otimes \omega_{Y/k})) \\ &\xrightarrow{s_{0*}} H^{m+n}(V, \mathcal{K}_{p+n}^{MW}(\pi^* \omega_{Y/k}^{-1} \otimes \omega_{V/k})) = H^{m+n}(V, \mathcal{K}_{p+n}^{MW}(\pi^* \det^{-1} V)). \end{aligned}$$

Definition 2.2. The Euler class $e(V) \in H^n(Y, \mathcal{K}_n^{MW}(\det^{-1} V))$ of V is the element $s_0^*(s_{0*}(1))$, where $1 \in H^0(Y, \mathcal{K}_0^{MW})$ is the section with constant value 1 in the sheaf of commutative rings \mathcal{K}_0^{MW} .

For Y smooth over k of pure dimension n , we have the tangent bundle T_Y and its Euler class $e(T_Y) \in H^n(Y, \mathcal{K}_n^{MW}(\omega_{Y/k}))$.

Definition 2.3. Let $p_X : X \rightarrow \text{Spec } k$ be a smooth projective variety of pure dimension n over k . The Chow-Witt Euler characteristic of X , $\chi^{CW}(X)$, is defined by

$$\chi^{CW}(X) := p_{X*}(e(T_X)) \in H^0(\text{Spec } k, \mathcal{K}_0^{MW}) = \text{GW}(k).$$

Remark 2.4. We have not checked that the pullback maps defined via Milnor-Witt cohomology agree with those defined by Fasel, which rely on the Gersten complex for their definition. Such an identification would follow by comparing the specialization map on Milnor-Witt cohomology with support with the specialization map on the Gersten complex, and we expect that the two pullback maps agree for an arbitrary morphism of smooth k -schemes.

It follows easily from the definitions, however, that the two pullback maps agree for a *smooth* morphism. As the Milnor-Witt cohomology is homotopy invariant, the pullback by a section s of a vector bundle $\pi : V \rightarrow Y$

$$s^* : H^a(V, \mathcal{K}_b^{MW}(\pi^* L)) \rightarrow H^a(Y, \mathcal{K}_b^{MW}(L))$$

is the inverse of $\pi^* : H^a(Y, \mathcal{K}_b^{MW}(L)) \rightarrow H^a(V, \mathcal{K}_b^{MW}(\pi^* L))$ for any line bundle L on Y , and thus Fasel's pullback s^* agrees with the sheaf-theoretic version used here. In particular, the Euler class as defined in Definition 2.2 agrees with Fasel's Euler class of a rank n vector bundle V , defined as $\tilde{c}_n(V)(1)$ (see [12, Definition 13.2.1]), which in turn agrees with the Euler class defined by Barge-Morel in [11, §2.1]. In fact, the definition of this last class does not use s_0^* at all, but just applies the inverse of the isomorphism π^* .

3. REVISITING THE GYSIN MAP

3.1. Purity isomorphism. The Gersten complex resolution of $\mathcal{K}_n^{MW}(L)$ gives rise to the following purity result.

Proposition 3.1 (Purity). *Let $i : W \rightarrow X$ be a codimension c closed immersion in \mathbf{Sm}/k . Let L be a line bundle on X and let N_i be the normal bundle of i . There is a isomorphism*

$$Ri^! \mathcal{K}_n^{MW}(L)_X \cong \mathcal{K}_{n-c}^{MW}(i^* L \otimes \det N_i)_W[-c],$$

natural with respect to smooth morphisms in \mathbf{Sm}/k .

Proof. We recall that the morphisms $i_* : Sh_{\text{Nis}}^{\mathbf{Ab}}(W) \rightarrow Sh_{\text{Nis}}^{\mathbf{Ab}}(X)$ and $i^* : Sh_{\text{Nis}}^{\mathbf{Ab}}(X) \rightarrow Sh_{\text{Nis}}^{\mathbf{Ab}}(W)$ are exact and $i^* i_* = \text{Id}$. Thus $Li^* = i^*$, $Ri_* = i_*$ and $i^* Ri_* = \text{Id}$. Thus it suffices to show that

$$i_* Ri^! \mathcal{K}_n^{MW}(L) \cong i_* \mathcal{K}_{n-c}^{MW}(i^* L \otimes \det N_i)[-c]$$

in $D^+ Sh_{\text{Nis}}^{\mathbf{Ab}}(X)$. Let $j : U := X \setminus W \rightarrow X$ be the inclusion. Applying the localization distinguished triangle

$$i_* Ri^! \rightarrow \text{Id} \rightarrow Rj_* j^*$$

to the Gersten resolution for $\mathcal{K}_n^{MW}(L)$, we see that $i_* Ri^! \mathcal{K}_n^{MW}(L)$ is represented by the subcomplex

$$\begin{aligned} \oplus_{x \in W(0)} i_{x*} K_{n-c}^{MW}(L \otimes \det \mathfrak{m}_{X,x} / \mathfrak{m}_{X,x}^2)(k(x)) \rightarrow \\ \dots \rightarrow \oplus_{x \in W(j)} i_{x*} K_{n-c-j}^{MW}(L \otimes \det \mathfrak{m}_{X,x} / \mathfrak{m}_{X,x}^2)(k(x)) \rightarrow \dots \end{aligned}$$

of the Gersten resolution for $\mathcal{K}_n^{MW}(L)$; here $\mathfrak{m}_{X,x}$ is the maximal ideal in the local ring $\mathcal{O}_{X,x}$. Note that this complex starts in degree c , and we have the canonical exact sequence

$$0 \rightarrow N_i^\vee \otimes k(x) \rightarrow \mathfrak{m}_{X,x} / \mathfrak{m}_{X,x}^2 \rightarrow \mathfrak{m}_{W,x} / \mathfrak{m}_{W,x}^2 \rightarrow 0,$$

so $i_* Ri^! \mathcal{K}_n^{MW}(L)$ is represented by $i_* \mathcal{C}^*(W, n-c, i^* L \otimes \det N_i)[-c]$, which is isomorphic in $D^+(Sh_{\text{Nis}}^{\mathbf{Ab}}(X))$ to $i_* \mathcal{K}_{n-c}^{MW}(i^* L \otimes \det N_i)[-c]$.

The naturality follows from the fact that the Gersten resolution is functorial with respect to pullback by smooth morphisms. \square

Corollary 3.2. *Let $p : V \rightarrow X$ be a rank r vector bundle on $X \in \mathbf{Sm}/k$ with 0-section $s_0 : X \rightarrow V$ and let L be a line bundle on X . There is an isomorphism*

$$Rs_0^! \mathcal{K}_n^{MW}(V, p^* L \otimes \det^{-1} V)_V \cong \mathcal{K}_{n-r}^{MW}(L)_X.$$

natural with respect to pullback by smooth morphisms. In particular,

$$Rs_0^! \mathcal{K}_r^{MW}(V, \det^{-1} V)_V \cong \mathcal{K}_{0X}^{MW}.$$

Proof. This follows from Proposition 3.1 and the canonical isomorphism $N_{s_0} \cong V$. \square

Remark 3.3. For a closed immersion $i : W \rightarrow X$ of k -schemes and a sheaf $\mathcal{F} \in Sh_{\text{Nis}}^{\mathbf{Ab}}(X)$, the localization distinguished triangle mentioned in the proof of Proposition 3.1 gives rise to the canonical isomorphism

$$H_W^m(X, \mathcal{F}) \cong H^m(W, Ri^! \mathcal{F}).$$

Thus, for $i : W \rightarrow X$ a codimension c closed immersion in \mathbf{Sm}/k and line bundle L on X , the purity isomorphism gives the isomorphism

$$H_W^m(X, \mathcal{K}_n^{MW}(L)) \cong H^{m-c}(W, \mathcal{K}_{n-c}^{MW}(i^* L \otimes \det N_i)).$$

For the case of the 0-section $s_0 : X \rightarrow V$ of a rank r vector bundle $p : V \rightarrow X$, we thus have the isomorphism

$$H_{s_0(X)}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V)) \cong H^0(X, \mathcal{K}_0^{MW})$$

and $\text{th}(V) \in H_{s_0(X)}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V))$ is the element corresponding to $1 \in H^0(X, \mathcal{K}_0^{MW})$.

3.2. The Thom class and the Thom isomorphism.

Definition 3.4. Let $\pi : V \rightarrow Y$ be a rank r vector bundle over $Y \in \mathbf{Sm}/k$ with 0-section $s_0 : Y \rightarrow V$. The *oriented Thom class* $\text{th}(V) \in H_{s_0(Y)}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V))$ is the element corresponding to $1 \in H^0(Y, \mathcal{K}_0^{MW})$ under the purity isomorphism

$$H_{s_0(Y)}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V)) \cong H^0(Y, \mathcal{K}_0^{MW}).$$

Theorem 3.5 (Thom isomorphism). *Let $\pi : V \rightarrow Y$ be a rank r vector bundle over $Y \in \mathbf{Sm}/k$ with 0-section $s_0 : Y \rightarrow V$, and let L be a line bundle on Y . Let $W \subset Y$ be a closed subset. Then the map*

$$\text{th}(V) \cup : H_W^a(Y, \mathcal{K}_b^{MW}(L \otimes \det V)) \rightarrow H_{s_0(W)}^{a+r}(V, \mathcal{K}_{b+r}^{MW}(p^* L))$$

sending $\alpha \in H_W^a(Y, \mathcal{K}_b^{MW}(L \otimes \det V))$ to $\text{th}(V) \cup p^ \alpha$ is an isomorphism.*

This is proven by Asok-Haesemeyer [5, Theorem 4.2.7] by a somewhat different argument.

Proof. The ring structure on \mathcal{K}_*^{MW} gives a multiplication on the Gersten resolution

$$\mathcal{C}^*(V, r, \det^{-1} V) \otimes \mathcal{K}_b^{MW}(L \otimes \det V)_Y \rightarrow \mathcal{C}^*(V, r + b, p^* L).$$

This passes to give a multiplication on the subcomplexes with support in $s_0(Y)$:

$$\mathcal{C}_{s_0(Y)}^*(V, r, \det^{-1} V) \otimes \mathcal{K}_b^{MW}(L \otimes \det V)_Y \rightarrow \mathcal{C}_{s_0(Y)}^*(V, r + b, p^* L).$$

If we have a Zariski hypercover $Y \rightarrow \mathcal{U}_*$, this multiplication extends to a multiplication

$$\begin{aligned} \mathcal{C}_{s_0(Y)}^*(V, r, \det^{-1} V)(V) \otimes \mathcal{K}_b^{MW}(L \otimes \det V)(\mathcal{U}_*) \\ \rightarrow \text{Tot}(\mathcal{C}_{s_0(Y)}^*(V, r + b, p^* L)(p^{-1}\mathcal{U}_*)) \end{aligned}$$

where $\mathcal{K}_b^{MW}(p^* L \otimes \det V)(\mathcal{U}_*)$ is the complex of sections over \mathcal{U}_* , and this in turn induces a multiplication

$$\begin{aligned} \mathcal{C}_{s_0(Y)}^*(V, r, \det^{-1} V)(V) \otimes \mathcal{K}_b^{MW}(L \otimes \det V)(\mathcal{U}_*)_W \\ \rightarrow \text{Tot}(\mathcal{C}_{s_0(Y)}^*(V, r + b, p^* L)(p^{-1}\mathcal{U}_*)_{p^{-1}W}) \end{aligned}$$

where

$$\begin{aligned} \mathcal{K}_b^{MW}(L \otimes \det V)(\mathcal{U}_*)_W \\ = \text{Cone}[\text{res} : \mathcal{K}_b^{MW}(L \otimes \det V)(\mathcal{U}_*) \rightarrow \mathcal{K}_b^{MW}(L \otimes \det V)(\mathcal{U}_* \times_Y (Y \setminus W))] \end{aligned}$$

and $\mathcal{C}_{s_0(Y)}^*(V, r + b, p^* L)(p^{-1}\mathcal{U}_*)_{p^{-1}W}$ is defined similarly.

Recalling that the Nisnevich and Zariski cohomology of the Milnor-Witt sheaves agree, if we take the cohomology of the complexes

$$\begin{aligned} \mathcal{C}_{s_0(Y)}^*(V, r, \det^{-1} V)(V), \\ \mathcal{K}_b^{MW}(L \otimes \det V)(\mathcal{U}_*)_W, \\ \text{Tot}(\mathcal{C}_{s_0(Y)}^*(V, r + b, p^* L)(p^{-1}\mathcal{U}_*)_{p^{-1}W}), \end{aligned}$$

and pass to the limit over the hypercover \mathcal{U}_* , we recover the cup product pairing

$$\begin{aligned} H_{s_0(Y)}^m(V, \mathcal{K}_r^{MW}(\det^{-1} V)) \otimes H_W^a(Y, \mathcal{K}_b^{MW}(L \otimes \det V)) \\ \rightarrow H_{s_0(Y) \cap p^{-1}W}^{m+a}(V, \mathcal{K}_{r+b}^{MW}(p^* L)) \end{aligned}$$

sending $\beta \otimes \alpha$ to $\beta \cup p^* \alpha$. On the other hand, we have isomorphisms

$$\begin{aligned} \mathcal{C}_{s_0(Y)}^*(V, r, \det^{-1} V)(V) &\cong \mathcal{C}^*(Y, 0, \mathcal{O}_Y)(Y)[-r] \\ \mathcal{C}_{s_0(Y)}^*(V, r + b, p^* L)(p^{-1}\mathcal{U}_*) &\cong \mathcal{C}^*(Y, b, L \otimes \det V)(\mathcal{U}_*)[-r], \\ \mathcal{C}_{s_0(Y)}^*(V, r + b, p^* L)(p^{-1}\mathcal{U}_* \times_Y (Y \setminus W)) \\ &\cong \mathcal{C}^*(Y \setminus W, b, L \otimes \det V)(\mathcal{U}_* \times_Y (Y \setminus W))[-r], \end{aligned}$$

as in the proof of Proposition 3.1, identifying the above cup product pairing with the cup product pairing

$$H^{m-r}(Y, \mathcal{K}_0^{MW}) \otimes H_W^a(Y, \mathcal{K}_b^{MW}(L \otimes \det V)) \rightarrow H_W^{m-r+a}(Y, \mathcal{K}_b^{MW}(L \otimes \det V)).$$

Since $\text{th}(Y) \in H_{s_0(Y)}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V))$ corresponds to $1 \in H^0(Y, \mathcal{K}_0^{MW})$, it follows that the composition

$$\begin{aligned} H_W^a(Y, \mathcal{K}_b^{MW}(L \otimes \det V)) &\xrightarrow{\text{th}(V) \cup} H_{s_0(W)}^{r+a}(V, \mathcal{K}_{r+b}^{MW}(p^* L)) \\ &\xrightarrow[\sim]{\text{purity}} H_W^a(Y, \mathcal{K}_b^{MW}(L \otimes \det V)) \end{aligned}$$

is the identity, and thus $\text{th}(V) \cup$ is an isomorphism. \square

Remark 3.6. The Thom class $\text{th}(V)$ is defined somewhat differently in the literature. For instance in [14, §2.4], the Koszul complex associated to the tautological section $t : \mathcal{O}_V \rightarrow \pi^* V$,

$$\text{Kos}(V) := \Lambda^r \pi^* V^\vee \rightarrow \dots \rightarrow \pi^* V^\vee \xrightarrow{t^\vee} \mathcal{O}_V,$$

has a canonical symmetric isomorphism

$$\phi_V : \text{Kos}(V) \rightarrow D_V(\text{Kos}(V)),$$

with respect to the duality $D_V(-) := \mathcal{H}om(-, \det^{-1} \pi^* V)[r]$ on the triangulated category $D_{s_0(Y)}^{perf}(V)$ of perfect complexes on V with support in $s_0(Y)$, with respect to the natural isomorphism $(-1)^{r(r+1)/2} \text{can} : \text{Id} \rightarrow D_V^2$. This gives an element in the Grothendieck-Witt group defined by Balmer-Walter ([10, Definition 1.4.3] and [38, §2]), $\text{GW}_0(D_{s_0(Y)}^{perf}(V), D_V, (-1)^{r(r+1)/2} \text{can})$. The devissage theorem [32, Theorem 6.1] gives an isomorphism

$$(3.1) \quad \text{GW}_0(D_{s_0(Y)}^{perf}(V), D_V, (-1)^{r(r+1)/2} \text{can}) \cong \text{GW}(Y)$$

We have the presheaf $U \mapsto \text{GW}(U)$ on Y_{Nis} with associated Nisnevich sheaf \mathcal{K}_0^{MW} and thus $(\text{Kos}(V), \phi_V)$ yields the element

$$(\widetilde{\text{Kos}(V)}, \phi_V) \in H^0(s_0(Y), \mathcal{K}_0^{MW}).$$

It turns out that $(\widetilde{\text{Kos}(V)}, \phi_V)$ is the unit section: this follows by restricting to an open subset of Y over which V is the trivial bundle and making an explicit computation. Just as for the construction given here, the purity isomorphism translates the unit section to an element

$$\text{th}(V) \in H_{s_0(V)}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V)),$$

which is the same as the class $\text{th}(V)$ constructed above, as both come from the unit section of \mathcal{K}_0^{MW} .

As the purity isomorphism is natural with respect to pullback by smooth maps, we have a similar naturality for the Thom isomorphism. To extend the functoriality of the Thom isomorphism to arbitrary morphisms, we give a purely cohomological description of the Thom class which does not rely on

the Gersten resolution in terms of Čech cohomology. We recall that, given a topological space T with a closed subspace A , a sheaf \mathcal{F} of abelian groups on T and an open cover $\mathcal{U} = \{U_\alpha\}$ of T , the Čech complex $\check{C}_A^*(\mathcal{U}, \mathcal{F})$ is the subcomplex of $\check{C}^*(\mathcal{U}, \mathcal{F})$ with

$$\check{C}_A^n(\mathcal{U}, \mathcal{F}) = \{s_{\alpha_*} \in \mathcal{F}(\cap_{i=0}^n U_{\alpha_i}) \mid s_{\alpha_*} = 0 \text{ if } U_{\alpha_i} \cap A = \emptyset \text{ for all } i\}.$$

Suppose we have a trivialization of our rank r vector bundle $V \rightarrow Y$ over some open $U \subset Y$,

$$\phi : V|_U \rightarrow \mathbb{A}_U^r = \text{Spec } \mathcal{O}_U[t_1, \dots, t_r]$$

Letting $V_{|U,i} \subset V|_U$ be the open subset $t_i \neq 0$, we have the open cover $\mathcal{V} := \{V_{|U,i}\}$ of $V|_U \setminus 0_U$. Passing from Čech cohomology to Nisnevich cohomology, the element $[\phi^* t_r][\phi^* t_{r-1}] \cdots [\phi^* t_1]$ of $K_r^{MW}(\cap_{i=1}^n V_{|U,i})$ together with the isomorphism $\det \phi : \det V|_U \rightarrow \mathcal{O}_U$ thus defines an element

$$[[\phi^* t_r][\phi^* t_{r-1}] \cdots [\phi^* t_1]] \otimes \phi^* t_1 \wedge \cdots \phi^* t_r \in H^{r-1}(V|_U \setminus 0_U, \mathcal{K}_r^{MW}(\det^{-1} V)).$$

By a standard computation, the boundary

$$\partial_{0_U}([[\phi^* t_r][\phi^* t_{r-1}] \cdots [\phi^* t_1]] \otimes (\phi^* t_1 \wedge \cdots \phi^* t_r)) \in H_{0_U}^r(V|_U, \mathcal{K}_r^{MW}(\det^{-1} V))$$

may be computed by applying the composition

$$\begin{aligned} & H^0(\cap_{i=1}^r V_{|U,i}, \mathcal{K}_r^{MW}(\det^{-1} V)) \\ & \xrightarrow{\partial} H^1_{(\phi^* t_r=0)}(\cap_{i=1}^{r-1} V_{|U,i}, \mathcal{K}_r^{MW}(\det^{-1} V)) \\ & \xrightarrow{\partial} H^2_{(\phi^* t_r=\phi^* t_{r-1}=0)}(\cap_{i=1}^{r-2} V_{|U,i}, \mathcal{K}_r^{MW}(\det^{-1} V)) \\ & \vdots \\ & \xrightarrow{\partial} H^r_{0_U}(V|_U, \mathcal{K}_r^{MW}(\det^{-1} V)) \end{aligned}$$

to $[[\phi^* t_r][\phi^* t_{r-1}] \cdots [\phi^* t_1]] \otimes (\phi^* t_1 \wedge \cdots \phi^* t_r)$. Identifying the various cohomologies with support with the corresponding terms in the Gersten complex, we follow the progress of $[[\phi^* t_r] \cdots [\phi^* t_1]] \otimes (\phi^* t_1 \wedge \cdots \phi^* t_r)$ as

$$\begin{aligned} & [[\phi^* t_r][\phi^* t_{r-1}] \cdots [\phi^* t_1]] \otimes \phi^* t_1 \wedge \cdots \phi^* t_r \\ & \quad \text{on } \cap_{i=1}^r V_{|U,i} \\ & \mapsto [[\phi^* t_{r-1}] \cdots [\phi^* t_1]] \otimes \partial/\partial \phi^* t_r \otimes \phi^* t_1 \wedge \cdots \phi^* t_r \\ & \quad \text{on } (\phi^* t_r = 0) \cap \cap_{i=1}^{r-1} V_{|U,i} \\ & \mapsto [[\phi^* t_{r-2}] \cdots [\phi^* t_1]] \otimes \partial/\partial \phi^* t_{r-1} \wedge \partial/\partial \phi^* t_r \otimes \phi^* t_1 \wedge \cdots \phi^* t_r \\ & \quad \text{on } (\phi^* t_r = \phi^* t_{r-1} = 0) \cap \cap_{i=1}^{r-2} V_{|U,i} \\ & \vdots \\ & \mapsto \partial/\partial \phi^* t_1 \wedge \cdots \wedge \partial/\partial \phi^* t_r \otimes \phi^* t_1 \wedge \cdots \phi^* t_r = 1 \text{ on } 0_U. \end{aligned}$$

Note that the resulting element in $H_{0_U}^r(V|_U, \mathcal{K}_r^{MW}(\det^{-1} V)) \cong \mathcal{K}_0^{MW}(U)$ is independent of the choice of trivialization ϕ .

Let $p^0 : V \setminus 0_Y \rightarrow Y$ be the restriction of the projection $p : V \rightarrow Y$. Define the sheaf $\bar{R}^r p_*^0 \mathcal{K}_r^{MW}(\det^{-1} V)$ on Y by

$$\begin{aligned} \bar{R}^r p_*^0 \mathcal{K}_r^{MW}(\det^{-1} V) \\ := \begin{cases} R^r p_*^0 \mathcal{K}_r^{MW}(\det^{-1} V)|_{V \setminus 0_Y} & \text{for } r > 0 \\ p_*^0 \mathcal{K}_r^{MW}(\det^{-1} V)|_{V \setminus 0_Y} / p_* \mathcal{K}_r^{MW}(\det^{-1} V)_V & \text{for } r = 0 \end{cases} \end{aligned}$$

By the homotopy invariance of Milnor-Witt cohomology, the boundary in the local cohomology sequence defines an isomorphism of sheaves on Y

$$\partial_{0_Y} : \bar{R}^{r-1} p_*^0 \mathcal{K}_r^{MW}(\det^{-1} V) \rightarrow R_{0_Y}^r p_* \mathcal{K}_r^{MW}(\det^{-1} V)$$

where $R_{0_Y}^r p_* \mathcal{K}_r^{MW}(\det^{-1} V)$ is the sheafification of the presheaf

$$U \mapsto H_{0_U}^r(V_U, \mathcal{K}_r^{MW}(\det^{-1} V)).$$

Thus, the locally defined elements

$$\begin{aligned} [[\phi^* t_r][\phi^* t_{r-1}] \cdots [\phi^* t_1]] \otimes (\phi^* t_1 \wedge \cdots \phi^* t_r) \\ \in H^{r-1}(V|_U \setminus 0_U, \mathcal{K}_n^{MW}(\det^{-1} V)) \end{aligned}$$

define a global section

$$\text{th}(V)^0 \in H^0(Y, \bar{R}^{r-1} p_*^0 \mathcal{K}_r^{MW}(\det^{-1} V))$$

with

$$\begin{aligned} (3.2) \quad \partial_{0_Y}(\text{th}(V)^0) &= \text{th}(V) \in H^0(Y, R_{0_Y}^r p_* \mathcal{K}_n^{MW}(\det^{-1} V)) \\ &= H_{0_Y}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V)). \end{aligned}$$

Proposition 3.7. 1. Let $f : Y \rightarrow X$ be a morphism in \mathbf{Sm}/k , and let $\tilde{f} : W \rightarrow V$ be a map of vector bundles over f . Then

$$\tilde{f}^* \text{th}(V) = \text{th}(W)$$

in $H_{s_0(Y)}^r(W, \mathcal{K}_r^{MW}(\det^{-1} W))$.

2. Suppose we have vector bundles $V_1 \rightarrow Y$, $V_2 \rightarrow Y$ of rank r_1 and r_2 , respectively. Let $p_i : V_1 \oplus V_2 \rightarrow V_i$ be the projections, $i = 1, 2$ and let $r = r_1 + r_2$. Then

$$\text{th}(V_1 \oplus V_2) = p_1^* \text{th}(V_1) \cup p_2^* \text{th}(V_2)$$

in $H_{s_0(Y)}^r(V_1 \oplus V_2, \mathcal{K}_r^{MW}(p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2))$.

Proof. For (1), suppose we have a trivialization of V over $U \subset X$, $\phi : V|_U \rightarrow \mathbb{A}_U^r$. This induces a trivialization of W over $f^{-1}(U)$, $\tilde{f}^* \phi : W|_{f^{-1}(U)} \rightarrow \mathbb{A}_{f^{-1}(U)}^r$, and

$$\begin{aligned} (\tilde{f}|_{f^{-1}(U)})^*([[\phi^* t_1][\phi^* t_2] \cdots [\phi^* t_n]] \otimes \phi^* t_1 \wedge \cdots \wedge \phi^* t_n) \\ = [[\tilde{f}^* \phi]^* t_1][(\tilde{f}^* \phi)^* t_2] \cdots [(\tilde{f}^* \phi)^* t_n] \otimes (\tilde{f}^* \phi)^* t_1 \wedge \cdots \wedge (\tilde{f}^* \phi)^* t_n \end{aligned}$$

Thus $\tilde{f}^*\text{th}(V)^0 = \text{th}(W)^0$ and hence

$$\tilde{f}^*\text{th}(V) = \tilde{f}^*\partial_{0_X}\text{th}(V)^0 = \partial_{0_Y}\text{th}(W)^0 = \text{th}(W).$$

For (2), let $s_0^2 : V_2 \rightarrow V_1 \oplus V_2$ be the 0-section to $p_2 : V_1 \oplus V_2 \rightarrow V_2$, let $p_{V_2} : V_2 \rightarrow Y$ be the projection and let $s_{0V_2} : Y \rightarrow V_2$ be the 0-section. Identifying $p_2 : V_1 \oplus V_2 \rightarrow V_2$ with the pullback vector bundle $p_{V_2}^* V_1 \rightarrow V_2$, the functoriality of the Thom class gives the identity

$$\text{th}(p_{V_2}^* V_1) = p_1^* \text{th}(V_1)$$

and we have the Thom isomorphism

$$\begin{aligned} H_{s_{0V_2}(Y)}^{r_2}(V_2, \mathcal{K}_{r_2}^{MW}(\det^{-1} V_2)) \\ \xrightarrow{p_1^* \text{th}(V_1) \cup} H_{s_0(Y)}^r(V_1 \oplus V_2, \mathcal{K}_r^{MW}(p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2)) \end{aligned}$$

inverse to the purity isomorphism

$$\begin{aligned} H_{s_0(Y)}^r(V_1 \oplus V_2, \mathcal{K}_r^{MW}(p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2)) \\ \rightarrow H_{s_{0V_2}(Y)}^{r_2}(V_2, \mathcal{K}_{r_2}^{MW}(\det^{-1} V_2)). \end{aligned}$$

Thus, the purity isomorphism sends $p_1^* \text{th}(V_1) \cup p_2^* \text{th}(V_2)$ to $\text{th}(V_2)$.

The above purity isomorphism is induced from the isomorphism of complexes

$$\begin{aligned} \mathcal{C}_{s_0^2(V_2)}^*(V_1 \oplus V_2, r, p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2)(V_1 \oplus V_2) \\ \cong \mathcal{C}^*(V_2, r_2, \det^{-1} V_2)(V_2) \end{aligned}$$

which restricts to the isomorphism of subcomplexes

$$\begin{aligned} \mathcal{C}_{s_0(Y)}^*(V_1 \oplus V_2, r, p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2)(V_1 \oplus V_2) \\ \cong \mathcal{C}_{s_{0V_2}(Y)}^*(V_2, r_2, \det^{-1} V_2)(V_2). \end{aligned}$$

As $\text{th}(V_2)$ is represented by $1 \in \mathcal{K}_0^{MW}(Y)$ on $s_{0V_2}(Y)$, the corresponding element of $\mathcal{C}_{s_0(Y)}^*(V_1 \oplus V_2, r, p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2)(V_1 \oplus V_2)$ is represented by $1 \in \mathcal{K}_0^{MW}(Y)$ on $s_0(Y)$. Therefore the same element in the complex $\mathcal{C}_{s_0^2(V_2)}^*(V_1 \oplus V_2, r, p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2)(V_1 \oplus V_2)$ represents both classes $p_1^* \text{th}(V_1) \cup p_2^* \text{th}(V_2)$ and $\text{th}(V_1 \oplus V_2)$, hence

$$p_1^* \text{th}(V_1) \cup p_2^* \text{th}(V_2) = \text{th}(V_1 \oplus V_2)$$

in $H_{s_0(Y)}^r(V_1 \oplus V_2, \mathcal{K}_r^{MW}(p_1^* \det^{-1} V_1 \otimes p_2^* \det^{-1} V_2))$. \square

Remarks 3.8. (1) The functoriality described in Proposition 3.7 implicitly includes the isomorphism $\det^{-1}(\tilde{f}^*) : f^* \det^{-1} V \rightarrow \det^{-1} W$.

(2) The map $\text{th}(-) \cup$ is natural: given a vector bundle map $\phi : W \rightarrow V$ over a morphism $f : Y \rightarrow X$, and a line bundle L on X , we have maps

$$(f, \det \phi)^* : H^m(X, \mathcal{K}_n^{MW}(L \otimes \det V)) \rightarrow H^m(Y, \mathcal{K}_n^{MW}(f^* L \otimes \det W)),$$

$$\phi^* : H_{s_0(X)}^{m+r}(V, \mathcal{K}_{n+r}^{MV}(p_V^* L)) \rightarrow H_{s_0(Y)}^{m+r}(W, \mathcal{K}_{n+r}^{MV}(p_W^* f^* L))$$

and a commutative diagram

$$\begin{array}{ccc} H^m(X, \mathcal{K}_n^{MW}(L \otimes \det V)) & \xrightarrow{(f, \det \phi)^*} & H^m(Y, \mathcal{K}_n^{MW}(f^* L \otimes \det V)) \\ \text{th}(V) \cup \downarrow & & \downarrow \text{th}(W) \cup \\ H_{s_0(X)}^{m+r}(V, \mathcal{K}_{n+r}^{MV}(p_V^* L)) & \xrightarrow{\phi^*} & H_{s_0(Y)}^{m+r}(W, \mathcal{K}_{n+r}^{MV}(p_W^* f^* L)) \end{array}$$

Here is another application of the cohomological description of the Thom class.

Proposition 3.9. *Let $V_1 \rightarrow Y$, $V_2 \rightarrow Y$ be vector bundles of rank r_1, r_2 , respectively. Let $\pi : V_1 \oplus V_2 \rightarrow Y$ be the Whitney sum, $s_0 : Y \rightarrow V_1 \oplus V_2$ the 0-section, $p_1 : V_1 \oplus V_2 \rightarrow V_1$, $p_2 : V_1 \oplus V_2 \rightarrow V_2$ the projections and $r = r_1 + r_2$. Then*

$$p_1^* \text{th}(V_1) \cup p_2^* \text{th}(V_2) = p_2^* \text{th}(V_2) \cup p_1^* \text{th}(V_1)$$

$$\text{in } H_{s_0(Y)}^r(V_1 \oplus V_2, \mathcal{K}_r^{MW}(\pi^* \det^{-1} V_1 \otimes \det^{-1} V_2)).$$

Proof. Denote $\pi^* \det^{-1} V_1 \otimes \det^{-1} V_2$ by \det_{12}^{-1} . Let $s_0^i : Y \rightarrow V_i$, $\tilde{s}_0^2 : V_1 \rightarrow V_1 \oplus V_2$, $\tilde{s}_0^1 : V_2 \rightarrow V_1 \oplus V_2$ be the 0-sections.

Since $H_{s_0(Y)}^r(V_1 \oplus V_2, \mathcal{K}_r^{MW}(\det_{12}^{-1})) \cong \mathcal{GW}(Y)$, it suffices to prove the identity after restriction to a dense open subscheme of Y . Changing notation, we may assume that V_1 and V_2 are trivial bundles; fix isomorphisms

$$V_1 \cong Y \times \text{Spec } k[x_1, \dots, x_{r_1}], \quad V_2 \cong Y \times \text{Spec } k[y_1, \dots, y_{r_2}]$$

inducing the isomorphism

$$V_1 \oplus V_2 \cong Y \times \text{Spec } k[x_1, \dots, x_{r_1}, y_1, \dots, y_{r_2}].$$

Let $V_{1i} := V_1 \setminus (x_i = 0)$, $V_{2j} := V_2 \setminus (y_j = 0)$, and set $V_{10} = V_1$, $V_{20} = V_2$. We consider x_i both as a unit on $\cap_{i=1}^{r_1} V_{1i}$ and as a section of V_1^\vee , and similarly for y_j . Letting $\mathcal{V}_1 = \{V_{10}, \dots, V_{1r_1}\}$, $\mathcal{V}_2 = \{V_{20}, \dots, V_{2r_2}\}$, $\partial \text{th}(V_1)^0$ is represented in Čech cohomology $H_{s_0^1(Y)}^{r_1}(\mathcal{V}_1, \mathcal{K}_{r_1}^{MW}(\det^{-1} V_1))$ by

$$[x_1] \cdots [x_{r_1}] \otimes x_1 \wedge \cdots \wedge x_{r_1} \text{ on } \cap_{i=0}^{r_1} V_{1i}$$

and $\partial \text{th}(V_2)^0$ is represented in Čech cohomology $H_{s_0^2(Y)}^{r_2}(\mathcal{V}_2, \mathcal{K}_{r_2}^{MW}(\det^{-1} V_2))$ by

$$[y_1] \cdots [y_{r_2}] \otimes y_1 \wedge \cdots \wedge y_{r_2} \text{ on } \cap_{j=0}^{r_2} V_{2j}.$$

Letting

$$\mathcal{V} = \{V_{10} \times_Y V_2, \dots, V_{1r_1} \times_Y V_2, V_1 \times_Y V_{20}, \dots, V_1 \times_Y V_{2r_2}\},$$

$p_1^* \text{th}(V_1) \cup p_2^* \text{th}(V_2)$ is represented in $H_{s_0(Y)}^r(\mathcal{V}, \mathcal{K}_r^{MW}(\det_{12}^{-1}))$ by

$$\begin{aligned} [x_1] \cdots [x_{r_1}] \cdot [y_1] \cdots [y_{r_2}] \otimes x_1 \wedge \cdots \wedge x_{r_1} \wedge y_1 \wedge \cdots \wedge y_{r_2} \\ \text{on } \cap_{i=0}^{r_1} V_{1i} \times_Y V_2 \cap V_1 \times_Y \cap_{j=0}^{r_2} V_{2j}. \end{aligned}$$

Thus $p_2^* \text{th}(V_2) \cup p_1^* \text{th}(V_1)$ is represented by

$$(-1)^{r_1 r_2} [y_1] \cdots [y_{r_2}] \cdot [x_1] \cdots [x_{r_1}] \otimes y_1 \wedge \cdots \wedge y_{r_2} \wedge x_1 \wedge \cdots \wedge x_{r_1} \\ \text{on } \cap_{i=0}^{r_1} V_{1i} \times_Y V_2 \cap V_1 \times_Y \cap_{j=0}^{r_2} V_{2j}.$$

Since

$$[y_1] \cdots [y_{r_2}] \cdot [x_1] \cdots [x_{r_1}] = (-\langle -1 \rangle)^{r_1 r_2} [x_1] \cdots [x_{r_1}] \cdot [y_1] \cdots [y_{r_2}],$$

$$y_1 \wedge \cdots \wedge y_{r_2} \wedge x_1 \wedge \cdots \wedge x_{r_1} = (-1)^{r_1 r_2} x_1 \wedge \cdots \wedge x_{r_1} \wedge y_1 \wedge \cdots \wedge y_{r_2},$$

and $a \otimes u \lambda = \langle u \rangle a \otimes \lambda$ in $\mathcal{K}_r^{MW}(\det_{12}^{-1})$, we have

$$(-1)^{r_1 r_2} [y_1] \cdots [y_{r_2}] \cdot [x_1] \cdots [x_{r_1}] \otimes y_1 \wedge \cdots \wedge y_{r_2} \wedge x_1 \wedge \cdots \wedge x_{r_1} \\ = [x_1] \cdots [x_{r_1}] \cdot [y_1] \cdots [y_{r_2}] \otimes x_1 \wedge \cdots \wedge x_{r_1} \wedge y_1 \wedge \cdots \wedge y_{r_2},$$

which proves the result. \square

4. REPRESENTING MILNOR-WITT COHOMOLOGY IN $\text{SH}(k)$

4.1. The homotopy t -structure and truncated co-homotopy. To relate the two Euler characteristics χ^{cat} and χ^{CW} , we will represent the cohomology $H^n(-, \mathcal{K}_n^{MW})$ and its twists in $\text{SH}(k)$, using Morel's homotopy t -structure. We follow the construction given in [5].

Recall that for a strictly \mathbb{A}^1 -invariant sheaf of abelian groups, M on \mathbf{Sm}/k , we have its \mathbb{G}_m -loops sheaf $M_{-1} := \mathcal{H}om(\mathbb{G}_m, M)$. The heart of the homotopy t -structure is equivalent to the category of *homotopy modules*, where a homotopy module (M_*, ϵ_*) is a sequence $M_* := (M_0, M_1, \dots)$ of strictly \mathbb{A}^1 -invariant sheaves of abelian groups together with isomorphisms $\epsilon_n : M_n \rightarrow (M_{n+1})_{-1}$. The functor $H_0 : \text{SH}(k) \rightarrow \text{SH}(k)^\heartsuit$ sends a T -spectrum \mathcal{E} to the homotopy module $H_0(\mathcal{E})_* = \pi_0(\mathcal{E})_{-*} := (\pi_{0,0}(\mathcal{E}), \dots, \pi_{-n,-n}(\mathcal{E}), \dots)$ with $\pi_0(\mathcal{E})_{-n} \rightarrow \mathcal{H}om(\mathbb{G}_m, \pi_0(\mathcal{E})_{-n-1})$ induced by the suspension isomorphism

$$[\Sigma_T^\infty U_+ \wedge \mathbb{G}_m, \Sigma_{\mathbb{G}_m}^{n+1} \mathcal{E}] \cong [\Sigma_T^\infty U_+, \Sigma_{\mathbb{G}_m}^n \mathcal{E}].$$

If \mathcal{E} is a commutative monoid in $\text{SH}(k)$, $H_0(\mathcal{E})$ receives an induced commutative monoid structure and if \mathcal{E} is t -non-negative, the canonical map $\mathcal{E} \rightarrow H_0(\mathcal{E})$ is monoidal.

We have the canonical isomorphism $(\mathcal{K}_{n+1}^{MW})_{-1} \cong \mathcal{K}_n^{MW}$, compatible via Morel's isomorphism $\pi_{-n,-n}(\mathbb{S}_k) \cong \mathcal{K}_n^{MW}$ with the canonical isomorphism $\pi_{-n-1,-n-1}(\mathbb{S}_k)_{-1} \cong \pi_{-n,-n}(\mathbb{S}_k)$ described above, giving description of the homotopy module $H_0(\mathbb{S}_k)_*$ as

$$H_0(\mathbb{S}_k)_* = \pi_0(\mathbb{S}_k)_{-*} \cong (\mathcal{K}_0^{MW}, \mathcal{K}_1^{MW}, \dots, \mathcal{K}_n^{MW}, \dots).$$

As a T -spectrum,

$$H_0(\mathbb{S}_k) = (K(\mathcal{K}_0^{MW}, 0), K(\mathcal{K}_1^{MW}, 1), \dots, K(\mathcal{K}_n^{MW}, n), \dots)$$

with bonding map $\Sigma_T K(\mathcal{K}_n^{MW}, n) \rightarrow K(\mathcal{K}_{n+1}^{MW}, n+1)$ the adjoint of the weak equivalence

$$\begin{aligned} K(\mathcal{K}_n^{MW}, n) &\cong K(\Omega_{\mathbb{G}_m} \mathcal{K}_{n+1}^{MW}, n) \\ &\cong \Omega_{S_1} \Omega_{\mathbb{G}_m} K(\mathcal{K}_{n+1}^{MW}, n+1) \cong \Omega_T K(\mathcal{K}_{n+1}^{MW}, n+1). \end{aligned}$$

Since \mathbb{S}_k is t -non-negative, there is a natural monoidal map $\tau_0 : \mathbb{S}_k \rightarrow H_0 \mathbb{S}_k$, which induces the identity map

$$\mathrm{GW}(k) = \mathbb{S}_k^{0,0}(k) \rightarrow (H_0 \mathbb{S}_k)^{0,0}(k) = K_0^{MW}(k) = \mathrm{GW}(k);$$

we may thus view $\chi^{cat}(X)$ as an element of $(H_0 \mathbb{S}_k)^{0,0}(k)$.

Since the model we have chosen for $H_0 \mathbb{S}_k$ is an Ω -spectrum, we have, for each $Y \in \mathbf{Sm}/k$ and each m , $0 \leq m \leq n$, the isomorphism

$$(4.1) \quad (H_0 \mathbb{S}_k)^{n+m,n}(Y) = [Y_+, \Omega_{S_1}^{n-m} K(\mathcal{K}_n^{MW}, n)]_{\mathcal{H}_\bullet(k)} = H_{Nis}^m(Y, \mathcal{K}_n^{MW}).$$

Similarly, one has $(H_0 \mathbb{S}_k)^{n+m,n}(Y) = 0$ for $m > n$. More generally, for $Z \subset Y$ a closed subset we have

$$(H_0 \mathbb{S}_k)^{m+n,n}(Y/(Y \setminus Z)) = H_Z^m(Y, \mathcal{K}_n^{MW}),$$

where H_Z^m is the Nisnevich cohomology with support in Z , and thus for $V \rightarrow Y$ a vector bundle with 0-section $s_0 : Y \rightarrow V$, we have the natural isomorphism

$$\theta_V : H_{s_0(Y)}^m(V, \mathcal{K}_n^{MW}) \rightarrow (H_0 \mathbb{S}_k)^{n+m,n}(\mathrm{Th}(V)).$$

Definition 4.1. Let $p : V \rightarrow Y$ be a rank r vector bundle with 0-section $s_0 : Y \rightarrow V$. Let

$$\vartheta_V : H^a(Y, \mathcal{K}_b^{MW}(\det V)) \rightarrow (H_0 \mathbb{S}_k)^{a+b+2r, b+r}(\mathrm{Th}(V))$$

be the isomorphism defined as the composition

$$\begin{aligned} H^a(Y, \mathcal{K}_b^{MW}(\det V)) &\xrightarrow[\sim]{\mathrm{th}(V) \cup} H_{s_0(Y)}^{a+r}(V, \mathcal{K}_{b+r}^{MW}) \\ &\xrightarrow[\sim]{\theta_V} (H_0 \mathbb{S}_k)^{a+b+2r, b+r}(\mathrm{Th}(V)). \end{aligned}$$

Remark 4.2. The map ϑ_V is natural in V . Let $\phi : V \rightarrow W$ be a vector bundle map over a morphism $f : Y \rightarrow X$. We have the map

$$\mathrm{Th}(\phi)^* : H_0(\mathbb{S}_k)^{a,b}(\mathrm{Th}(W)) \rightarrow H_0(\mathbb{S}_k)^{a,b}(\mathrm{Th}(V))$$

with $\mathrm{Th}(\phi)^* \circ \theta_W = \theta_V \circ \phi^*$, so we have a similar naturality for ϑ_V :

$$\mathrm{Th}(\phi)^* \circ \vartheta_W = \vartheta_V \circ (f, \det \phi)^*.$$

Note that $\det \phi$ might not be the identity even if $\det V$ and $\det W$ are trivial line bundles.

Lemma 4.3. *Let $\pi : V \rightarrow Y$ be a rank r vector bundle, take integers $0 \leq m \leq n$ and let O_Y denote the trivial line bundle on Y . We have the canonical isomorphism $\mathrm{Th}(O_Y \oplus V) \cong \Sigma_T \mathrm{Th}(V)$; let*

$$\Sigma_T : H_0(\mathbb{S}_k)^{n+m+2r,n+r}(\mathrm{Th}(V)) \rightarrow H_0(\mathbb{S}_k)^{n+m+2r+2,n+r+1}(\mathrm{Th}(O_Y \oplus V))$$

be the associated suspension isomorphism. Then the diagram

$$\begin{array}{ccc} H^m(Y, \mathcal{K}_n^{MW}(\det V)) & & \\ \vartheta_V \downarrow & \searrow \vartheta_{O_Y \oplus V} & \\ H_0(\mathbb{S}_k)^{n+m+2r,n+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_T} & H_0(\mathbb{S}_k)^{n+m+2r+2,n+r+1}(\mathrm{Th}(O_Y \oplus V)) \end{array}$$

commutes.

Proof. Let $p_1 : O_Y \oplus V \rightarrow O_Y$, $p_2 : O_Y \oplus V \rightarrow V$ be the projections. We have the identity of Proposition 3.7(2)

$$\mathrm{th}(O_Y \oplus V) = p_1^*(\mathrm{th}(O_Y)) \cup p_2^*(\mathrm{th}(V))$$

in $H_{s_0(Y)}^{r+1}(O_Y \oplus V, \mathcal{K}_{r+1}^{MW}(p_1^* \det^{-1} O_Y \otimes p_2^* \det^{-1} V))$. Thus we may write $\mathrm{th}_{O_Y \oplus V} \cup (-)$ as

$$\mathrm{th}_{O_Y \cup V} \cup (\alpha) = p_1^*(\mathrm{th}_{O_Y}) \cup p_2^*(\mathrm{th}_V \cup \pi^* \alpha).$$

This reduces us to showing that the diagram

$$\begin{array}{ccc} H_{s_0(Y)}^{m+r}(V, \mathcal{K}_{n+r}^{MV}) & \xrightarrow{p_1^*(\mathrm{th}_{O_Y}) \cup p_2^*(-)} & H_{s_0(Y)}^{m+r+1}(O_Y \oplus V, \mathcal{K}_{n+r+1}^{MV}) \\ \theta_V \downarrow & & \downarrow \theta_{O_Y \oplus V} \\ H_0(\mathbb{S}_k)^{n+m+2r,n+r}(\mathrm{Th}(V)) & \xrightarrow{\Sigma_T} & H_0(\mathbb{S}_k)^{n+m+2r+2,n+r+1}(\mathrm{Th}(O_Y \oplus V)) \end{array}$$

commutes.

Let $\mathbf{1} \in H_0(\mathbb{S}_k)^{0,0}(k)$ be the element representing the canonical map $\tau : \mathbb{S}_k \rightarrow H_0(\mathbb{S}_k)$ and let $\mathbf{th} \in H_0(\mathbb{S}_k)^{2,1}(T)$ be the image of $\mathbf{1}$ under the suspension isomorphism. Then for an arbitrary space $\mathcal{X} \in \mathbf{Spc}(k)$, the suspension isomorphism $H_0(\mathbb{S}_k)^{a,b}(\mathcal{X}) \rightarrow H_0(\mathbb{S}_k)^{a+2,b+1}(\Sigma_T \mathcal{X}_+)$ is the composition

$$H_0(\mathbb{S}_k)^{a,b}(\mathcal{X}) \xrightarrow{p_2^*} H_0(\mathbb{S}_k)^{a,b}(\mathbb{A}^1 \times \mathcal{X}) \xrightarrow{p_1^* \mathbf{th} \cup} H_0(\mathbb{S}_k)^{a+2,b+1}(T \wedge \mathcal{X}_+)$$

Thus we need only show that $\theta_{O_Y}(\mathrm{th}_{O_Y}) = p_Y^*(\mathbf{th})$. By the naturality of θ_{O_Y} and the functoriality of th_{O_Y} (both in Y), we reduce to the case $Y = \mathrm{Spec} k$.

In this case, $O_Y = \mathbb{A}_k^1 = \mathrm{Spec} k[t]$ as a line bundle over $\mathrm{Spec} k$. We have the isomorphism

$$H^0(\mathbb{G}_m, \mathcal{K}_1^{MW}) \xrightarrow{\delta} H_0^1(\mathbb{A}^1, \mathcal{K}_1^{MW}).$$

arising from the cohomology sequence of the triple $(\{1\}, \mathbb{A}^1 \setminus \{0\}, \mathbb{A}^1)$.

Let $[t] \in K_1^{MW}(\mathbb{G}_m)$ be the element corresponding to the unit t on $\mathbb{A}^1 \setminus \{0\}$. Then the identity (3.2) shows $\delta([t]) = \mathrm{th}(\mathbb{A}^1)$ in $H_0^1(\mathbb{A}^1, \mathcal{K}_1^{MW})$.

We have the commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{G}_m, \mathcal{K}_1^{MW}) & \xrightarrow{\delta} & H_0^1(\mathbb{A}^1, \mathcal{K}_1^{MW}) \\ \alpha \downarrow & & \downarrow \theta_{\mathbb{A}^1} \\ H_0(\mathbb{S}_k)^{1,1}(\mathbb{G}_m) & \xrightarrow{\delta} & H_0(\mathbb{S}_k)^{2,1}(T) \end{array}$$

where α is the canonical isomorphism and the bottom map δ arises from the cofiber sequence in $\mathbf{Spc}_\bullet(k)$, $\mathbb{G}_m \rightarrow (\mathbb{A}^1, 1) \rightarrow T$. Since $\alpha([t])$ is the image of $\mathbf{1}$ under the suspension isomorphism $H_0(\mathbb{S}_k)^{0,0}(k) \cong H_0(\mathbb{S}_k)^{1,1}(\mathbb{G}_m)$, it follows that $\theta_{\mathbb{A}^1}(\mathrm{th}(\mathbb{A}^1)) = \mathrm{th}$, as desired. \square

Corollary 4.4. *Suppose we have an isomorphism $\psi : O_Y^r \rightarrow V$. Let $\alpha_\psi : H^m(Y, \mathcal{K}_n^{MW}(\det V)) \rightarrow H_0(\mathbb{S}_k)^{m+n,n}(Y)$ be the composition*

$$\begin{aligned} H^m(Y, \mathcal{K}_n^{MW}(\det V)) & \xrightarrow{(\mathrm{Id}, \det \psi)^*} H^m(Y, \mathcal{K}_n^{MW}(\det O_Y^r)) \\ & = H^m(Y, \mathcal{K}_n^{MW}) \xrightarrow{\sim} H_0(\mathbb{S}_k)^{m+n,n}(Y) \end{aligned}$$

where the equality arises from the canonical isomorphism $\det O_Y^r \cong O_Y$ and the last map is the isomorphism (4.1). Then via the identity $\mathrm{Th}(O_Y^r) = \Sigma_T^r Y_+$, the diagram

$$\begin{array}{ccc} H^m(Y, \mathcal{K}_n^{MW}(\det V)) & \xrightarrow{\vartheta_V} & H_0(\mathbb{S}_k)^{m+n+2r,n+r}(\mathrm{Th}(V)) \\ \alpha_\psi \downarrow & & \downarrow \mathrm{Th}(\psi)^* \\ H_0(\mathbb{S}_k)^{m+n,n}(Y) & \xrightarrow{\Sigma_T^r} & H_0(\mathbb{S}_k)^{m+n+2r,n+r}(\Sigma_T^r Y_+) \end{array}$$

commutes, where Σ_T^r is the suspension isomorphism.

Proof. In case $V = O_Y^r$ and ψ is the identity, this follows directly from Lemma 4.3. The general case follows from this and the commutative diagram

$$\begin{array}{ccc} H^m(Y, \mathcal{K}_n^{MW}(\det V)) & \xrightarrow{\vartheta_V} & H_0(\mathbb{S}_k)^{m+n+2r,n+r}(\mathrm{Th}(V)) \\ (\mathrm{Id}, \det \psi)^* \downarrow & & \downarrow \mathrm{Th}(\psi)^* \\ H^m(Y, \mathcal{K}_n^{MW}(\det O_Y^r)) & \xrightarrow{\vartheta_{O_Y^r}} & H_0(\mathbb{S}_k)^{m+n+2r,n+r}(\mathrm{Th}(O_Y^r)), \end{array}$$

discussed in Remark 4.2. \square

4.2. Duality and twisted Milnor-Witt cohomology. Let $X \in \mathbf{Sm}/k$ have dimension d_X over k . As X is quasi-projective, we may fix a locally closed immersion $i_X : X \rightarrow \mathbb{P}_k^d$ as in §1.1; we retain the notation of that section. In particular, we have the Jouanolou cover $p_X : \tilde{X} \rightarrow X$ and the vector bundle $\tilde{\nu}_{\tilde{X}} \rightarrow \tilde{X}$ on \tilde{X} . Letting $N = d^2 + 2d$, $\tilde{\nu}_{\tilde{X}}$ has rank $N - d_X$.

Via the canonical isomorphism (1.7) $\det \tilde{\nu}_{\tilde{X}} \cong p_X^* \omega_{X/k}$ we have the Thom isomorphism

$$\mathrm{th}(\tilde{\nu}_{\tilde{X}}) \cup : H^m(\tilde{X}, \mathcal{K}_n^{MW}(p_X^* \omega_{X/k})) \rightarrow H_{\tilde{s}_0(\tilde{X})}^{m+N-d_X}(\tilde{\nu}_{\tilde{X}}, \mathcal{K}_{n+N-d_X}^{MW})$$

where $\tilde{s}_0 : \tilde{X} \rightarrow \tilde{\nu}_{\tilde{X}}$ is the zero section. In addition, we have the comparison isomorphism

$$\theta_{\tilde{\nu}_{\tilde{X}}} : H_{\tilde{s}_0(\tilde{X})}^{m+N-d_X}(\tilde{\nu}_{\tilde{X}}, \mathcal{K}_{n+N-d_X}^{MW}) \rightarrow H_0(\mathbb{S}_k)^{m+n+2N-2d_X, n+N-d_X}(\mathrm{Th}(\tilde{\nu}_{\tilde{X}})).$$

In case X is projective, the dual $\Sigma_T^\infty X_+^\vee$ is canonically isomorphic to $\Sigma_T^{-N} \Sigma_T^\infty \mathrm{Th}(\tilde{\nu}_{\tilde{X}})$; in the quasi-projective case, we nonetheless may consider the object $\Sigma_T^{-N} \Sigma_T^\infty \mathrm{Th}(\tilde{\nu}_{\tilde{X}})$ of $\mathrm{SH}(k)$, which we have denoted by $\Sigma_T^\infty X_+^{\tilde{\vee}}$, recognizing that $\Sigma_T^\infty X_+^{\tilde{\vee}}$ may depend on the various choices we made in its construction. See however Remark 1.2.

The map $\theta_{\tilde{\nu}_{\tilde{X}}}$ induces the isomorphism

$$\theta_{\tilde{\nu}_{\tilde{X}}}^s : H_{\tilde{s}_0(\tilde{X})}^{m+N-d_X}(\tilde{\nu}_{\tilde{X}}, \mathcal{K}_{n+N-d_X}^{MW}) \rightarrow H_0(\mathbb{S}_k)^{m+n-2d_X, n-d_X}(\Sigma_T^\infty X_+^{\tilde{\vee}}).$$

By the homotopy invariance of twisted Milnor-Witt cohomology,³ the pull-back

$$p_X^* : H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k})) \rightarrow H^m(\tilde{X}, \mathcal{K}_n^{MW}(p_X^* \omega_{X/k}))$$

is an isomorphism. Thus, we have the comparison isomorphism $\theta_X^\vee := \theta_{\tilde{\nu}_{\tilde{X}}}^s \circ \mathrm{th}(\tilde{\nu}_{\tilde{X}}) \cup \circ p_X^*$,

$$(4.2) \quad \theta_X^\vee : H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k})) \rightarrow H_0(\mathbb{S}_k)^{m+n-2d_X, n-d_X}(\Sigma_T^\infty X_+^{\tilde{\vee}}),$$

which is actually an isomorphism

$$\theta_X^\vee : H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k})) \rightarrow H_0(\mathbb{S}_k)^{m+n-2d_X, n-d_X}(\Sigma_T^\infty X_+^\vee)$$

in case X is projective.

More generally, suppose we have a rank r vector bundle V on X . As above, the homotopy invariance of twisted Milnor-Witt cohomology combined with the appropriate Thom isomorphism gives us the isomorphism

$$\begin{aligned} & H^a(X, \mathcal{K}_b^{MW}(\omega_{X/k} \otimes \det V)) \\ & \xrightarrow{\vartheta_{\tilde{\nu}_{\tilde{X}} \oplus p_X^* V}} H^0(\mathbb{S}_k)^{a+b+2N+2r-2d_X, b+N+r-d_X}(\mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)). \end{aligned}$$

Define

$$\Sigma_T^\infty(X; V)_+^{\tilde{\vee}} := \Sigma_T^{-d^2-2d-r} \Sigma_T^\infty \mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V).$$

Stabilizing and desuspending $\vartheta_{\tilde{\nu}_{\tilde{X}} \oplus p_X^* V}$ gives the isomorphism

$$\theta_{X,V}^\vee : H^a(X, \mathcal{K}_b^{MW}(\omega_{X/k} \otimes \det V)) \rightarrow H^0(\mathbb{S}_k)^{a+b-2d_X, b-d_X}(\Sigma_T^\infty(X; V)_+^{\tilde{\vee}}).$$

To make the notation uniform, we allow the case of the rank zero vector bundle $0 \rightarrow X$ and set $\Sigma_T^\infty(X; 0)_+^{\tilde{\vee}} = \Sigma_T^\infty X_+^{\tilde{\vee}}$ and $\theta_{X,0}^\vee = \theta_X^\vee$.

³The homotopy invariance follows from the fact that the sheaves \mathcal{K}_n^{MW} are strictly \mathbb{A}^1 -invariant; see [26, §2.2] for a proof of this.

5. PUSHFORWARD MAPS IN MILNOR-WITT COHOMOLOGY

5.1. Pushforward in Milnor-Witt cohomology: the case of a closed immersion. As mentioned above, Fasel has defined, for $f : Y \rightarrow X$ a proper morphism in \mathbf{Sm}/k of relative dimension d , a pushforward map

$$f_* : H^m(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes f^* L)) \rightarrow H^{m-d}(X, \mathcal{K}_{n-d}^{MW}(\omega_{X/k} \otimes L))$$

for each line bundle L on X . This is accomplished via the Gersten complexes $C^*(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes f^* L))$, $C^*(X, \mathcal{K}_{n-d}^{MW}(\omega_{X/k} \otimes L))$, which are respective flasque resolutions of $\mathcal{K}_n^{MW}(\omega_{Y/k} \otimes f^* L)$ and $\mathcal{K}_{n-d}^{MW}(\omega_{X/k} \otimes L)$, and an explicit map of complexes (see [12, Corollaire 10.4.5], where the map is denoted $(f_*)_G$)

$$C(f)_* : C^*(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes f^* L)) \rightarrow C^*(X, \mathcal{K}_{n-d}^{MW}(\omega_{X/k} \otimes L))[-d].$$

Fasel's pushforward maps are functorial (this follows from [12, Proposition 2.3.3, Proposition 8.3.5]; see [12, §10] for further details).

Here we will define functorial pushforward maps in Milnor-Witt cohomology which are more directly adapted to the comparison isomorphism (4.2). We will show that our pushforward maps agree with Fasel's for projective morphisms. Until we show that the two definitions agree, we will denote Fasel's pushforward map for a proper morphism f by f_*^ϕ .

Our pushforward maps in Milnor-Witt cohomology for a closed immersion are defined using the oriented Thom classes and homotopy purity.

Let $\iota : Y \rightarrow X$ be a codimension r closed immersion in \mathbf{Sm}/k and let $\pi : N_\iota \rightarrow Y$ be the normal bundle with zero-section s_0 . Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the kernel of the surjection $\iota^* : \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Y$. $\mathcal{I}_Y/\mathcal{I}_Y^2$ is a locally free \mathcal{O}_Y -module of rank r ; the normal bundle N_ι is the vector bundle on Y associated to the dual $(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$.

The map

$$\iota_* : H^n(Y, \mathcal{K}_n^{MW}(\omega_{Y/k})) \rightarrow H^{n+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k}))$$

is defined as the composition

$$\begin{aligned} H^n(Y, \mathcal{K}_n^{MW}(\omega_{Y/k})) &\xrightarrow{\text{th}(N_\iota) \cup} H_{s_0(Y)}^{n+r}(N_\iota, \mathcal{K}_{n+r}^{MW}(\pi^*(\omega_{Y/k} \otimes \det^{-1} N_\iota))) \\ &\cong H_{s_0(Y)}^{n+r}(N_\iota, \mathcal{K}_{n+r}^{MW}(\pi^* \iota^* \omega_{X/k})) \cong H_{\iota(Y)}^{n+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k})) \\ &\rightarrow H^{n+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k})) \end{aligned}$$

The first isomorphism is induced by the canonical isomorphism of line bundles $\omega_{Y/k} \otimes \det^{-1} N_\iota \cong \iota^* \omega_{X/k}$ arising from the exact sequence

$$0 \rightarrow \mathcal{I}_Y/\mathcal{I}_Y^2 \rightarrow \iota^* \Omega_{X/k} \rightarrow \Omega_{Y/k} \rightarrow 0,$$

and the second is by homotopy purity or equivalently, by identification of the appropriate Gersten complexes. The last map is “forget supports”.

More generally, the same construction yields a pushforward map for cohomology twisted by a line bundle L on X :

$$(5.1) \quad \iota_* : H^n(Y, \mathcal{K}_n^{MW}(\omega_{Y/K} \otimes \iota^* L)) \rightarrow H^{n+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/K} \otimes L))$$

Having defined pushforward for a closed immersion, we recall the definition of the Euler class:

Definition 5.1. Let $p : V \rightarrow X$ be a rank r vector bundle on $X \in \mathbf{Sm}/k$ with 0-section $s_0 : X \rightarrow V$. The Euler class $e(V)$ is the element $s_{0*} s_{0*}^*(1_X) \in H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$.

Let $Z \subset X$ be a closed subset and let $s : X \rightarrow V$ be a section with $s^{-1}(0_X) \subset Z$. The *Euler class with support*

$$e_Z(V; s) \in H_Z^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$$

is defined as $e_Z(V; s) := s^*(\text{th}(V))$.

As $s_{0*}(1_X)$ is by definition the image in $H^r(V, \mathcal{K}_r^{MW}(\det^{-1} V))$ of $\text{th}(V) \in H_{0X}^r(V, \mathcal{K}_r^{MW}(\det^{-1} V))$, it follows that $e(V)$ is the image of $e_X(V; s_0)$ under the identification

$$H_X^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)) = H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)).$$

Furthermore, it follows from the homotopy invariance of Milnor-Witt cohomology that, for an arbitrary section s of V and closed subset $Z \subset X$ containing the zero-locus of s , $e(V)$ is the image of $e_Z(V; s)$ under the “forget supports” map

$$H_Z^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)) \rightarrow H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)).$$

Remark 5.2. The reader should however be aware that the local Euler class $e_Z(V; s)$ depends in general on the choice of s : if s' is another section with zero-locus contained in Z , the family of sections $\tilde{s} := t \cdot s + (1-t)s'$ of $p_X^* V$ on $X \times \mathbb{A}^1$ will not in general have support contained in $Z \times \mathbb{A}^1$. If however \tilde{s} *does* have support contained in $Z \times \mathbb{A}^1$, then the homotopy invariance of Milnor-Witt cohomology gives the identity $e_Z(V; s) = e_Z(V; s')$. Taking $Z = X$ and $s' = s_0$ gives the identity $e(V) = \text{im}(e_Z(V; s))$ in $H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$ asserted above.

Proposition 5.3. *The map (5.1) agrees with Fasel’s pushforward map ι_*^ϕ*

Proof. Both maps factor canonically through the “forget supports” map

$$H_{\iota(Y)}^{n+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/K} \otimes L)) \rightarrow H^{n+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/K} \otimes L))$$

so we need only check that the maps

$$\iota_*, \iota_*^\phi : H^n(Y, \mathcal{K}_n^{MW}(\omega_{X/K} \otimes L)) \rightarrow H_{\iota(Y)}^{n+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/K} \otimes L))$$

agree. The Milnor-Witt cohomology is \mathbb{A}^1 -homotopy invariant; using the deformation to the normal bundle reduces us to the case of the closed immersion given by the 0-section $s_0 : Y \rightarrow V$ for a rank r vector bundle

$p : V \rightarrow Y$. In this case, s_{0*} is given by

$$\mathrm{th}(V) \cup : H^n(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \iota^* L)) \rightarrow H_{s_0(Y)}^{n+r}(V, \mathcal{K}_{n+r}^{MW}(\omega_{V/K} \otimes L)),$$

noting that $\omega_{V/K} = p^*(\det^{-1} V)$. As we have seen in the proof of Theorem 3.5, the Thom isomorphism $\mathrm{th}(V) \cup$ is the map on cohomology induced by the purity isomorphism of Gersten complexes

$$C^*(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \iota^* L)) \cong C_{s_0(Y)}^*(V, \mathcal{K}_{n+r}^{MW}(\omega_{V/K} \otimes L))[r].$$

This is exactly the map $C(s_0)_*$, where

$$C(s_0)_* : C^*(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \iota^* L)) \rightarrow C_{s_0(Y)}^*(V, \mathcal{K}_{n+r}^{MW}(\omega_{V/K} \otimes L))[r]$$

is the isomorphism of complexes used to define

$$s_{0*}^\phi : H^n(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \iota^* L)) \rightarrow H_{s_0(Y)}^{n+r}(V, \mathcal{K}_{n+r}^{MW}(\omega_{V/K} \otimes L)).$$

□

In the next few results, we prove some useful properties of our pushforward maps and Euler classes.

Lemma 5.4. *Let $\iota : Y \rightarrow X$ be a codimension r closed immersion in \mathbf{Sm}/k , let $g : X' \rightarrow X$ be a morphism in \mathbf{Sm}/k , transverse to ι , giving the cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{\iota'} & X' \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{\iota} & X \end{array}$$

with $\iota' : Y' \rightarrow X'$ a codimension r closed immersion in \mathbf{Sm}/k . Let W be a closed subset of Y , Z a closed subset of X with $\iota(W) \subset Z$, let $W' = g'^{-1}(W)$, $Z' = g^{-1}(Z)$. Let L be a line bundle on X . Then the diagram

$$\begin{array}{ccc} H_W^m(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \iota^* L)) & \xrightarrow{\iota_*} & H_Z^{m+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k} \otimes L)) \\ g'^* \downarrow & & \downarrow g^* \\ H_{W'}^m(Y', \mathcal{K}_n^{MW}(g'^*(\omega_{Y'/k} \otimes \iota'^* L))) & \xrightarrow{\iota'_*} & H_{Z'}^{m+r}(X', \mathcal{K}_{n+r}^{MW}(g^*(\omega_{X/k} \otimes L))) \end{array}$$

commutes.

Proof. We first note that $\iota'^* \omega_{X'/X} = \omega_{Y'/Y}$, so the bottom pull-back ι'_* is well-defined by taking the twisting with respect to $g^* L \otimes \omega_{X'/X}^{-1}$.

The desired commutativity is a consequence of the following points (we omit the line bundle L to lighten the notation).

- Let $\tilde{g} : N_{\iota'} \rightarrow N_{\iota}$ be the map induced by (g, g') . Since the square is transversal, we have a commutative square

$$\begin{array}{ccc} H_{s_0(Z)}^{m+r}(N_{\iota}, \mathcal{K}_{n+r}^{MW}(\pi^* \iota^*(\omega_{X/k}))) & \xrightarrow{\sim} & H_{\iota(Z)}^{m+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k})) \\ \tilde{g}^* \downarrow & & \downarrow g^* \\ H_{s_0(Z')}^{m+r}(N_{\iota'}, \mathcal{K}_{n+r}^{MW}(\pi'^* \iota'^* g'^*(\omega_{X/k}))) & \xrightarrow{\sim} & H_{\iota'(Z')}^{m+r}(X', \mathcal{K}_{n+r}^{MW}(g^*(\omega_{X/k}))) \end{array}$$

where the horizontal arrows are the purity isomorphisms. This in turn follows by noting that we have canonical isomorphisms of the deformation diagrams used to define the purity isomorphisms for ι and ι'

$$X' \times_X \text{Def}(\iota) \cong \text{Def}(\iota').$$

See Remark 1.6 for the details on deformation diagrams.

- The functoriality of the Thom class (Proposition 3.7(1)) gives rise to a commutative diagram

$$\begin{array}{ccc} H^n(Y, \mathcal{K}_n^{MW}(\omega_{Y/k})) & \xrightarrow{\text{th}(N_{\iota}) \cup} & H_{s_0(Y)}^{n+r}(N_{\iota}, \mathcal{K}_{n+r}^{MW}(\pi^* \iota^*(\omega_{X/k}))) \\ g'^* \downarrow & & \downarrow \tilde{g}^* \\ H^m(Y', \mathcal{K}_n^{MW}(g'^*(\omega_{Y/k}))) & \xrightarrow{\text{th}(N_{\iota'}) \cup} & H_{s_0(Y')}^{n+r}(N_{\iota'}, \mathcal{K}_{n+r}^{MW}(\pi'^* \iota'^* g'^*(\omega_{X/k}))). \end{array}$$

□

Lemma 5.5. *For $j = 1, 2$, let $\iota_j : Y_j \rightarrow X_j$ be a codimension r_j closed immersion in \mathbf{Sm}/k , let $Z_j \subset X_j$ be a closed subset of X_j and let L_j be a line bundle on X_j . Let $W_j = \iota_j^{-1}(Z_j)$ and let α_j be an element of $H_{W_j}^{m_j}(Y_j, \mathcal{K}_{n_j}^{MW}(\iota_j^* L_j))$. Let $L_1 \boxtimes L_2 = p_1^* L_1 \otimes p_2^* L_2$, let $\alpha_1 \boxtimes \alpha_2$ denote the element*

$$p_1^* \alpha_1 \cup p_2^* \alpha_2 \in H_{W_1 \times_k W_2}^{m_1+m_2}(Y_1 \times_k Y_2, \mathcal{K}_{n_1+n_2}^{MW}((\iota_1 \times \iota_2)^*(L_1 \boxtimes L_2)))$$

and define

$$\iota_{1*}(\alpha_1) \boxtimes \iota_{2*}(\alpha_2) \in H_{Z_1 \times Z_2}^{m_1+m_2+r_1+r_2}(X_1 \times_k X_2, \mathcal{K}_{n_1+n_2+r_1+r_2}^{MW}(L_1 \boxtimes L_2))$$

similarly. Then

$$(\iota_1 \times \iota_2)_*(\alpha_1 \boxtimes \alpha_2) = \iota_{1*}(\alpha_1) \boxtimes \iota_{2*}(\alpha_2).$$

Proof. By factoring $\iota_1 \times \iota_2$ as $(\iota_1 \circ \text{Id}_{X_2}) \circ (\text{Id}_{Y_1} \times \iota_2)$ and using symmetry, we reduce to the case $\iota_2 = \text{Id}_{X_2}$; we write $r_1 = r$, $\iota_1 = \iota$, and so on. Noting that the deformation diagrams⁴ for ι and $\iota \times \text{Id}_{X_2}$ satisfy

$$\text{Def}(\iota) \times_k X_2 = \text{Def}(\iota \times \text{Id}_{X_2})$$

⁴see Remark 1.6 for the notation

we reduce to the case in which ι is the 0-section $s_0 : X_1 \rightarrow N$ for some rank r vector bundle $p : N \rightarrow X_1$. Thus $\iota \times \text{Id}_{X_2}$ is the 0-section $s_0^2 : X_1 \times X_2 \rightarrow p_1^*N = N \times_k X_2$.

In this case, we have

$$\text{th}(p_1^*N) = p_1^*\text{th}(N)$$

by Proposition 3.7(1) and thus

$$\begin{aligned} (\iota_1 \times \iota_2)_*(\alpha_1 \boxtimes \alpha_2) &= s_{0*}^2(\alpha_1 \boxtimes \alpha_2) \\ &= \text{th}(p_1^*N) \cup (p \times \text{Id}_{X_2})^*(\alpha_1 \boxtimes \alpha_2) \\ &= p_1^*\text{th}(N) \cup p_1^*(p^*\alpha) \cup p_2^*\alpha_2 \\ &= p_1^*(\text{th}(N) \cup p^*\alpha) \cup p_2^*\alpha_2 \\ &= s_{0*}(\alpha) \boxtimes \alpha_2. \end{aligned}$$

□

Proposition 5.6. *Let $\iota : Y \rightarrow X$ be a codimension r closed immersion in \mathbf{Sm}/k , let Z_1, Z_2 be closed subsets of X and let L_1, L_2 be line bundles on X . Take $x \in H_{Z_1}^{n_1}(X, \mathcal{K}_m^{MW}(L_1))$, $y \in H_{\iota^{-1}(Z_2)}^{n_2}(Y, \mathcal{K}_{m_2}^{MW}(\omega_{Y/k} \otimes \iota^*L_2))$. Then*

$$\iota_*(\iota^*(x) \cup y) = x \cup \iota_*(y)$$

in $H_{Z_1 \cap Z_2}^{n_1+n_2+r}(X, \mathcal{K}_{m_1+m_2}^{MW}(\omega_{X/k} \otimes L_1 \otimes L_2))$.

Proof. Let $\delta_X : X \rightarrow X \times_k X$ and $\delta_Y : Y \rightarrow Y \times_k Y$ be the diagonal maps, let $W_i = \iota^{-1}(Z_i)$ and let $M_i = \iota^*L_i$, $i = 1, 2$. For classes $\alpha \in H_{Z_1}^{n_1}(X, \mathcal{K}_m^{MW}(L_1))$, $\beta \in H_{Z_2}^{n_2}(X, \mathcal{K}_{m_2+r}^{MW}(\omega_{X/k} \otimes L_2))$ we have

$$\alpha \cup \beta = \delta_X^*(\alpha \boxtimes \beta).$$

Similarly

$$\alpha \cup \beta = \delta_Y^*(\alpha \boxtimes \beta)$$

for classes $\alpha \in H_{W_1}^{n_1}(Y, \mathcal{K}_m^{MW}(\iota^*L_1))$, $\beta \in H_{W_2}^{n_2}(Y, \mathcal{K}_{m_2}^{MW}(\iota^*L_2))$.

Applying Lemma 5.4 to the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{(\iota, \text{Id}_Y)} & X \times_k Y \\ \downarrow \iota & & \downarrow \text{Id}_X \times \iota_Y \\ X & \xrightarrow{\delta_X} & X \times_k X \end{array}$$

and using Lemma 5.5, we have

$$\begin{aligned} \iota_*(\iota^*(x) \cup y) &= \iota_*\delta_Y^*(\iota^*(x) \boxtimes y) \\ &= \iota_*\delta_Y^*(\iota \times \text{Id}_Y)^*(x \boxtimes y) \\ &= \iota_*(\iota, \text{Id}_Y)^*(x \boxtimes y) \\ &= \delta_X^* \circ (\text{Id}_X \times \iota)_*(x \boxtimes y) \\ &= \delta_X^*(x \boxtimes \iota_*y) \\ &= x \cup \iota_*y \end{aligned}$$

□

Lemma 5.7. *Let V_1, V_2 be vector bundles on $X \in \mathbf{Sm}/k$ of respective ranks r_1, r_2 . Then*

$$e(V_1 \oplus V_2) = e(V_1) \cup e(V_2)$$

in $H^{r_1+r_2}(X, \mathcal{K}_{r_1+r_2}^{MW}(\det^{-1} V_1 \otimes \det^{-1} V_2))$. If we have sections $s_1 : X \rightarrow V_1$, $s_2 : X \rightarrow V_2$ and closed subsets Z_i of X with $s_i^{-1}(0_X) \subset Z_i$, $i = 1, 2$, then $s := (s_1, s_2) : X \rightarrow V_1 \oplus V_2$ has $s^{-1}(0_X) \subset Z_1 \cap Z_2$ and

$$e_{Z_1 \cap Z_2}(V_1 \oplus V_2; s) = e_{Z_1}(V_1; s_1) \cup e_{Z_2}(V_2; s_2)$$

in $H_{Z_1 \cap Z_2}^{r_1+r_2}(X, \mathcal{K}_{r_1+r_2}^{MW}(\det^{-1} V_1 \otimes \det^{-1} V_2))$.

Proof. This all follows directly from the identity (Proposition 3.7(2))

$$p_1^* \text{th}(V_1) \cup p_2^* \text{th}(V_2) = \text{th}(V_1 \oplus V_2).$$

□

Lemma 5.8. *Let $\iota : Y \rightarrow X$ be a codimension c closed immersion in \mathbf{Sm}/k and let $p : E \rightarrow X$ be a rank r vector bundle on X with $s : X \rightarrow E$. Let $E_Y \rightarrow Y$ be the pull-back bundle $\iota^* E$ with induced section $s_Y : Y \rightarrow E_Y$. Let $Z = s^{-1}(0_E)$, $W = (s_Y)^{-1}(0_{E_Y})$. Suppose there is a rank c vector bundle V on X with a section $t : X \rightarrow V$, transverse to the 0-section $s_0 : X \rightarrow V$ and with zero-locus Y . Letting N_ι denote the normal bundle to ι , the section t defines an isomorphism*

$$\det t : \iota^* \det V \rightarrow \det N_\iota$$

and thereby an isomorphism

$$\det t : \iota^*(\omega_{X/k} \otimes \det V) \rightarrow \omega_{Y/k}.$$

This in turn gives us the push-forward map

$$(\iota, \det t)_* : H_W^r(Y, \mathcal{K}_r^{MW}(\iota^* L)) \rightarrow H_Z^{r+c}(X, \mathcal{K}_r^{MW}(\mathbb{L} \otimes \det^{-1} V))$$

for each line bundle L on X . Then

$$(5.2) \quad (\iota, \det t)_*(e_W(E_Y; s_Y)) = e_Z(E; s) \cup e_Y(V; t)$$

in $H_W^{r+c}(X, \mathcal{K}_{r+c}^{MW}(\det^{-1} E \otimes \det^{-1} V))$.

Before we prove the lemma, we note that the existence of a pair (V, t) for $Y \subset X$ as above is always satisfied if Y has codimension one on X , namely $V = \mathcal{O}_X(Y)$ with canonical section t .

Proof of Lemma 5.8. We have the canonical isomorphism

$$\omega_{Y/k} \cong \pi^*(\omega_{X/k} \otimes \det^{-1} V).$$

Letting $L = \omega_{X/k}^{-1}$, Lemma 5.4 applied to the transverse cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & X \\ \downarrow \iota & & \downarrow s_0 \\ X & \xrightarrow{t} & V \end{array}$$

gives us the commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{K}_0^{MW}) & \xrightarrow{s_{0*}} & H_{s_0(X)}^c(V, \mathcal{K}_c^{MW}(\pi^* \det^{-1} V)) \\ \downarrow \iota^* & & \downarrow t^* \\ H^0(Y, \mathcal{K}_0^{MW}) & \xrightarrow{(\iota, \det t)_*} & H_Y^c(X, \mathcal{K}_c^{MW}(\det^{-1} V)). \end{array}$$

Thus

$$(\iota, \det t)_*(1_Y) = (\iota, \det t)_*(\iota^*(1_X)) = t^*(s_{0*}(1_X)) := e_Y(V; t).$$

Using the projection formula Proposition 5.6, this gives

$$\begin{aligned} (\iota, \det t)_*(e_W(E_Y; s_Y)) &= (\iota, \det t)_*(\iota^* e_Z(E; s) \cup 1_Y) \\ &= e_Z(E; s) \cup (\iota, \det t)_*(1_Y) \\ &= e_Z(E; s) \cup e_Y(V; t). \end{aligned}$$

□

5.2. Pushforward and duality. One can also define pushforward maps in twisted $H_0(\mathbb{S}_k)$ -cohomology by using the objects $\Sigma_T^\infty X_+^{\tilde{V}}$.

Let $\iota : Y \rightarrow X$ as above be a codimension r closed immersion, with X quasi-projective. Choose an embedding $i_X : X \rightarrow \mathbb{P}^d$ as in §1.1, giving the closed embedding $i_Y := i_X \circ \iota : Y \rightarrow \mathbb{P}^d$. Retaining the notation of §1.1 and extending this in the evident manner to Y , we have the cartesian diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\iota}} & \tilde{X} \\ p_Y \downarrow & & \downarrow p_X \\ Y & \xrightarrow{\iota} & X; \end{array}$$

composing $\tilde{\iota}$ with the zero section $s_0 : \tilde{X} \rightarrow \tilde{\nu}_{\tilde{X}}$ gives the map $\tilde{s}_0 := s_0 \circ \tilde{\iota} : \tilde{Y} \rightarrow \tilde{\nu}_{\tilde{X}}$. Since \tilde{Y} is affine, we have the isomorphism of the normal bundle $N_{\tilde{s}_0}$ with $N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}})$, which in turn is isomorphic to $\tilde{\nu}_{\tilde{Y}}$; these isomorphisms are unique up to choices in splitting extensions, hence induce an isomorphism

$$\mathrm{Th}(N_{\tilde{s}_0}) \cong \mathrm{Th}(\tilde{\nu}_{\tilde{Y}}),$$

unique up to \mathbb{A}^1 -homotopy. Composing this with the quotient map

$$\mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \rightarrow \tilde{\nu}_{\tilde{X}}/(\tilde{\nu}_{\tilde{X}} \setminus \tilde{s}_0(\tilde{Y}))$$

and the purity isomorphism

$$\tilde{\nu}_{\tilde{X}}/(\tilde{\nu}_{\tilde{X}} \setminus \tilde{s}_0(\tilde{Y})) \cong \mathrm{Th}(N_{\tilde{s}_0})$$

gives the map

$$\iota_{X \subset \mathbb{P}^d}^{\tilde{\vee}} : \mathrm{Th}(\tilde{\nu}_{\tilde{X}}) \rightarrow \mathrm{Th}(\tilde{\nu}_{\tilde{Y}}).$$

Stabilizing and applying $\Sigma_T^{-d^2-2d}$ gives the map

$$\iota^{\tilde{\vee}} : \Sigma_T^\infty X_+^{\tilde{\vee}} \rightarrow \Sigma_T^\infty Y_+^{\tilde{\vee}}$$

in $\mathrm{SH}(k)$.

More generally, suppose we have a rank r vector bundle V on X . An analogous quotient construction gives the map

$$\iota_{X \subset \mathbb{P}^d; V}^{\tilde{\vee}} : \mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \rightarrow \mathrm{Th}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \iota^* V).$$

in $\mathcal{H}_\bullet(k)$, which we may stabilize and apply $\Sigma_T^{-d^2-2d-r}$ to give the map

$$\iota_V^{\tilde{\vee}} : \Sigma_T^\infty(X; V)_+^{\tilde{\vee}} \rightarrow \Sigma_T^\infty(Y; \iota^* V)_+^{\tilde{\vee}}$$

in $\mathrm{SH}(k)$.

Lemma 5.9. *Suppose $X \subset \mathbb{P}^d$ is projective, let $\iota : Y \rightarrow X$ be a closed immersion in \mathbf{Sm}/k and let $\iota^\vee : \Sigma_T^\infty X_+^\vee \rightarrow \Sigma_T^\infty Y_+^\vee$ be the dual of $\Sigma^\infty \iota : \Sigma_T^\infty Y_+ \rightarrow \Sigma_T^\infty X_+$. Then $\iota^\vee = \iota^{\tilde{\vee}}$.*

Proof. As X and Y are projective, we have $\Sigma_T^\infty X_+^\vee = \Sigma_T^\infty X_+^{\tilde{\vee}}$ and $\Sigma_T^\infty X_+^\vee = \Sigma_T^\infty X_+^{\tilde{\vee}}$. The dual of $\Sigma^\infty \iota$ is defined as the composition

$$\begin{aligned} \Sigma_T^\infty X_+^\vee &\cong \Sigma_T^\infty X_+^\vee \wedge \mathbb{S}_k \xrightarrow{\mathrm{Id} \wedge \delta_Y} \Sigma_T^\infty X_+^\vee \wedge \Sigma_T^\infty Y_+ \wedge \Sigma_T^\infty Y_+^\vee \\ &\xrightarrow{\mathrm{Id} \wedge \Sigma^\infty \iota \wedge \mathrm{Id}} \Sigma_T^\infty X_+^\vee \wedge \Sigma_T^\infty X_+ \wedge \Sigma_T^\infty Y_+^\vee \\ &\xrightarrow{ev_X \wedge \mathrm{Id}} \mathbb{S}_k \wedge \Sigma_T^\infty Y_+^\vee \cong \Sigma_T^\infty Y_+^\vee. \end{aligned}$$

We claim that the diagram

$$(5.3) \quad \begin{array}{ccc} \Sigma_T^\infty X_+^\vee & & \\ \downarrow \wr & & \\ \Sigma_T^\infty X_+^\vee \wedge \mathbb{S}_k & \xrightarrow{(\mathrm{Id} \wedge \Sigma^\infty \iota \wedge \mathrm{Id}) \circ (\mathrm{Id} \wedge \delta_Y)} & \Sigma_T^\infty X_+^\vee \wedge \Sigma_T^\infty X_+ \wedge \Sigma_T^\infty Y_+^\vee \\ \mathrm{Id} \wedge \delta_X \downarrow & & \downarrow \mathrm{Id} \wedge \mathrm{Id} \wedge \iota^{\tilde{\vee}} \\ \Sigma_T^\infty X_+^\vee \wedge \Sigma_T^\infty X_+ \wedge \Sigma_T^\infty X_+^\vee & \xrightarrow{\mathrm{Id} \wedge \mathrm{Id} \wedge \iota^{\tilde{\vee}}} & \Sigma_T^\infty X_+^\vee \wedge \Sigma_T^\infty X_+ \wedge \Sigma_T^\infty Y_+^\vee \\ \downarrow ev_X \wedge \mathrm{Id} & & \downarrow ev_X \wedge \mathrm{Id} \\ \mathbb{S}_k \wedge \Sigma_T^\infty X_+^\vee & \xrightarrow{\mathrm{Id} \wedge \iota^{\tilde{\vee}}} & \mathbb{S}_k \wedge \Sigma_T^\infty Y_+^\vee \\ \downarrow \wr & & \downarrow \wr \\ \Sigma_T^\infty X_+^\vee & \xrightarrow{\iota^{\tilde{\vee}}} & \Sigma_T^\infty Y_+^\vee \end{array}$$

commutes. Indeed, the only question is the commutativity of the upper triangle. Noting that $\text{Id}_X \times \iota \circ \Delta_Y$ is the (transposed) graph morphism (ι, Id_Y) , we reduce to showing the commutativity of

$$\begin{array}{ccc} T^{d^2+2d} & \xrightarrow{\eta_{X \subset \mathbb{P}^d}} & \text{Th}(\tilde{\nu}_{\tilde{X}}) \\ & \searrow \eta_{Y \subset \mathbb{P}^d} & \downarrow \iota_{X \subset \mathbb{P}^d}^{\tilde{\nu}} \\ & & \text{Th}(\tilde{\nu}_{\tilde{Y}}) \end{array}$$

in $\mathcal{H}_\bullet(k)$. Since both $\eta_{X \subset \mathbb{P}^d}$ and $\eta_{Y \subset \mathbb{P}^d}$ factor through $\tilde{\eta}_{\mathbb{P}^d} : T^{d^2+2d} \rightarrow \text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d})$, we need only show the commutativity of

$$(5.4) \quad \begin{array}{ccc} \text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d}) & \xrightarrow{\pi_{X \subset \mathbb{P}^d}} & \text{Th}(\tilde{\nu}_{\tilde{X}}) \\ & \searrow \pi_{Y \subset \mathbb{P}^d} & \downarrow \iota_{X \subset \mathbb{P}^d}^{\tilde{\nu}} \\ & & \text{Th}(\tilde{\nu}_{\tilde{Y}}) \end{array}$$

where $\pi_{X \subset \mathbb{P}^d} : \text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d}) \rightarrow \text{Th}(\tilde{\nu}_{\tilde{X}})$ is the composition of the quotient map

$$\text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d}) \rightarrow \text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d}) / [\text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d}) \setminus s_0(i_{\tilde{X}}(\tilde{X}))]$$

with the purity isomorphism

$$\text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d}) / [\text{Th}_{\mathbb{P}^d}(\tilde{\nu}_{\mathbb{P}^d}) \setminus s_0(i_{\tilde{X}}(\tilde{X}))] \cong \text{Th}_{\tilde{X}}(\tilde{\nu}_{\tilde{X}}),$$

and $\pi_{Y \subset \mathbb{P}^d}$ is defined similarly.

Since $\iota_{X \subset \mathbb{P}^d}^{\tilde{\nu}}$ is defined by a similar quotient map followed by a purity isomorphism, the commutativity of (5.4) follows from our comments on the compatibility of the purity isomorphism with closed immersions in Remark 1.6.

As the composition along the left-hand column in (5.3) is the identity and along the right-hand side one has ι^\vee , this proves the lemma. \square

Lemma 5.10. *Let $\iota : Y \rightarrow X$ be a codimension r closed immersion of smooth equidimensional quasi-projective k -schemes, with X a locally closed subscheme of \mathbb{P}^d for some d . Let V be a rank s vector bundle on X and let $d_Y = \dim_k Y$. Then the diagram*

$$\begin{array}{ccc} H^m(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \iota^* \det V)) & \xrightarrow{\iota_*} & H^{m+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k} \otimes \det V)) \\ \theta_{Y, \iota^* V}^\vee \downarrow \wr & & \wr \downarrow \theta_{X, V}^\vee \\ H_0(\mathbb{S}_k)^{m+n-2d_Y, n-d_Y}(\Sigma_T^\infty(Y; \iota^* V)_+^{\tilde{\vee}}) & \xrightarrow[\iota^{\vee*}]{} & H_0(\mathbb{S}_k)^{m+n-2d_Y, n-d_Y}(\Sigma_T^\infty(X; V)_+^{\tilde{\vee}}) \end{array}$$

commutes.

Proof. Let $D_Y = d^2 + 2d + s - d_Y$ and let $\bar{V} = \iota^* V$. Using the definition of $\theta_{X, V}^\vee$ and $\theta_{Y, \iota^* V}^\vee$, the statement of the lemma is the same as asserting the

commutativity of the outer square in the diagram

$$\begin{array}{ccc}
H^m(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \det \bar{V})) & \xrightarrow{\iota_*} & H^{m+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k} \otimes \det V)) \\
\downarrow p_Y^* \wr & & \downarrow \wr p_X^* \\
H^m(\tilde{Y}, \mathcal{K}_n^{MW}(p_Y^*(\omega_{Y/k} \otimes \det \bar{V}))) & \xrightarrow{\tilde{\iota}_*} & H^{m+r}(\tilde{X}, \mathcal{K}_{n+r}^{MW}(p_X^*(\omega_{X/k} \otimes \det V))) \\
\downarrow \vartheta_{\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}} \wr & & \downarrow \wr \vartheta_{\tilde{\nu}_{\tilde{X}} \oplus p_X^* V} \\
H_0(\mathbb{S}_k)^{m+n+2D_Y, n+D_Y}(\mathrm{Th}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V})) & \xrightarrow{\iota_{X \subset \mathbb{P}^d, V}^{\tilde{\vee}}} & H_0(\mathbb{S}_k)^{m+n+2D_Y, n+D_Y}(\mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \\
\parallel & & \parallel \\
H_0(\mathbb{S}_k)^{m+n-2d_Y, n-d_Y}(\Sigma_T^\infty(Y, \bar{V})_+^{\tilde{\vee}}) & \xrightarrow{\iota_V^{\tilde{\vee}}} & H_0(\mathbb{S}_k)^{m+n-2d_Y, n-d_Y}(\Sigma_T^\infty(X, V)_+^{\tilde{\vee}}).
\end{array}$$

The top square commutes by Lemma 5.4. The bottom square commutes by the definition of $\iota_V^{\tilde{\vee}}$ as $\Sigma_T^{-d^2-2d-s} \Sigma_T^\infty \iota_{X \subset \mathbb{P}^d, V}^{\tilde{\vee}}$.

For the middle square, we recall that the map $\iota_{X \subset \mathbb{P}^d, V}^{\tilde{\vee}}$ factors as the map on cohomology induced by the composition of the isomorphism

$$\mathrm{Th}(\phi) : \mathrm{Th}(N_i \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \rightarrow \mathrm{Th}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V})$$

with the purity isomorphism

$$\mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) / (\mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \setminus \tilde{s}_0 \circ i_{\tilde{Y}}(\tilde{Y})) \xrightarrow{\sim} \mathrm{Th}(N_i \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)),$$

followed by the “forget supports” map.

Similarly, the map $\tilde{\iota}_*$ factors as the composition of the Thom isomorphism

$$\begin{aligned}
& H^m(\tilde{Y}, \mathcal{K}_n^{MW}(p_Y^*(\omega_{Y/k} \otimes \det \bar{V}))) \\
& \xrightarrow{\mathrm{th}(N_i) \cup} H_{s_0(\tilde{Y})}^{m+r}(N_i, \mathcal{K}_{n+r}^{MW}(\pi^* p_Y^*(\omega_{Y/k} \otimes \det \bar{V}) \otimes \det^{-1} N_i))
\end{aligned}$$

with the map on cohomology induced by the purity isomorphism $\tilde{X}/\tilde{X} \setminus \tilde{Y} \cong \mathrm{Th}(N_i)$ and the “forget supports” map.

The map $\vartheta_{\tilde{\nu}_{\tilde{X}} \oplus p_X^* V}$ lifts canonically to a map on cohomology with supports

$$\begin{aligned}
& H_{\tilde{\iota}(\tilde{Y})}^{m+r}(\tilde{X}, \mathcal{K}_{n+r}^{MW}(p_X^*(\omega_{X/k} \otimes \det V))) \\
& \xrightarrow{\vartheta_{\tilde{\nu}_{\tilde{X}} \oplus p_X^* V}^Y} H_{s_0 \circ \tilde{\iota}(\tilde{Y})}^{m+n+2D_Y, n+D_Y}(\mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)),
\end{aligned}$$

where s_0 is the zero section of $\tilde{\nu}_{\tilde{X}} \oplus p_X^* V$. As the maps $\tilde{\iota}_*$ and $\iota_{X \subset \mathbb{P}^d, V}^{\tilde{\vee}}$ by their definition factor through the respective “forget supports” map, we may

replace cohomology with cohomology with supports in the middle diagram to prove its commutativity. This gives us the diagram

(5.5)

$$\begin{array}{ccc}
H^m(\tilde{Y}, \mathcal{K}_n^{MW}(p_Y^*(\omega_{Y/k} \otimes \det \bar{V}))) & \xrightarrow{\tilde{\iota}_*} & H_{\tilde{\iota}(\tilde{Y})}^{m+r}(\tilde{X}, \mathcal{K}_{n+r}^{MW}(p_X^*(\omega_{X/k} \otimes \det V))) \\
\downarrow \vartheta_{\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}} \wr & & \downarrow \vartheta_{\tilde{\nu}_{\tilde{X}} \oplus p_X^* V} \wr \\
H_0(\mathbb{S}_k)^{m+n+2D_Y, n+D_Y}(\mathrm{Th}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V})) & \xrightarrow{\iota_{X \subset \mathbb{P}^d, V}^{\tilde{\nu}}} & H_0(\mathbb{S}_k)_{s_0 \circ \tilde{\iota}(\tilde{Y})}^{m+n+2D_Y, n+D_Y}(\mathrm{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V))
\end{array}$$

whose commutativity we need to verify.

We have the diagram

(A)

$$\begin{array}{ccc}
H^m(\tilde{Y}, \mathcal{K}_n^{MW}(p_Y^*(\omega_{Y/k} \otimes \det \bar{V}))) & \xrightarrow{\mathrm{th}(N_{\tilde{\iota}}) \cup} & H_{s_0(\tilde{Y})}^{m+r}(N_{\tilde{\iota}}, \mathcal{K}_{n+r}^{MW}(\pi^* p_Y^*(\omega_{Y/k} \otimes \det \bar{V}) \otimes \det^{-1} N_{\tilde{\iota}})) \\
\downarrow \mathrm{th}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}) \cup \wr & & \downarrow \wr \mathrm{th}(\pi^*(\tilde{\iota}^* \tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \cup \\
H_{s'_0(\tilde{Y})}^{m+D_Y}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}, \mathcal{K}_{n+D_Y}^{MW}) & \xrightarrow[\phi^*]{\sim} & H_{s''_0(\tilde{Y})}^{m+D_Y}(N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V), \mathcal{K}_{n+D_Y}^{MW}).
\end{array}$$

where $\tilde{\iota} : \tilde{Y} \rightarrow \tilde{X}$ is the inclusion, $\pi : N_{\tilde{\iota}} \rightarrow \tilde{Y}$ is the normal bundle of $\tilde{\iota}$, and s'_0 and s''_0 are the respective zero-sections of $\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}$ and $N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)$. The isomorphism $\phi : N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \rightarrow \tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}$ is induced by a choice of a splitting in the exact sequence

$$0 \rightarrow T_{\tilde{Y}} \rightarrow \tilde{\iota}^* T_{\tilde{X}} \rightarrow N_{\tilde{\iota}} \rightarrow 0;$$

and the canonical isomorphism $\tilde{\iota}^* p_X^* V \cong p_Y^* \bar{V}$. By the \mathbb{A}^1 -invariance of Milnor-Witt cohomology, ϕ^* is independent of the choice of splitting.

We have the following additional diagrams:

(B)

$$\begin{array}{ccc}
H_{s'_0(\tilde{Y})}^{m+D_Y}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}, \mathcal{K}_{n+D_Y}^{MW}) & \xrightarrow[\phi^*]{\sim} & H_{s''_0(\tilde{Y})}^{m+D_Y}(N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V), \mathcal{K}_{n+D_Y}^{MW}) \\
\downarrow \theta_{\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}} \wr & & \downarrow \theta_{N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)} \wr \\
H_0(\mathbb{S}_k)^{m+n+2D_Y, n+D_Y}(\mathrm{Th}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V})) & \xrightarrow[\mathrm{Th}(\phi)^*]{\sim} & H_0(\mathbb{S}_k)^{m+n+2D_Y, n+D_Y}(\mathrm{Th}(N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)))
\end{array}$$

(C)

$$\begin{array}{ccc}
H_{s_0(\tilde{Y})}^{m+r}(N_{\tilde{t}}, \mathcal{K}_{n+r}^{MW}(\pi^* p_Y^*(\omega_{Y/k} \otimes \det \bar{V}) \otimes \det^{-1} N_{\tilde{t}})) & & \\
\downarrow \text{th}(\pi^* \tilde{t}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \cup & \searrow \sim & H_{\tilde{t}(\tilde{Y})}^{m+r}(\tilde{X}, \mathcal{K}_{n+r}^{MW}(p_X^*(\omega_{X/k} \otimes \det V))) \\
H_{s_0''(\tilde{Y})}^{m+D_Y}(N_{\tilde{t}} \oplus \tilde{t}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V), \mathcal{K}_{n+D_Y}^{MW}) & & \downarrow \text{th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \cup \\
& \searrow \sim & H_{s_0 \tilde{t}(\tilde{Y})}^{m+D_Y}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V, \mathcal{K}_{n+D_Y}^{MW})
\end{array}$$

(D)

$$\begin{array}{ccc}
H_{s_0''(\tilde{Y})}^{m+D_Y}(N_{\tilde{t}} \oplus \tilde{t}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V), \mathcal{K}_{n+D_Y}^{MW}) & & \\
\downarrow \theta_{N_{\tilde{t}} \oplus \tilde{t}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)} & \searrow \sim & H_{s_0 \tilde{t}(\tilde{Y})}^{m+D_Y}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V, \mathcal{K}_{n+D_Y}^{MW}) \\
H_0(\mathbb{S}_k)^{m+n+2D_Y, n+D_Y}(\text{Th}(N_{\tilde{t}} \oplus \tilde{t}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V))) & & \downarrow \theta_{\tilde{\nu}_{\tilde{X}} \oplus p_X^* V} \\
& \searrow \sim & H_0(\mathbb{S}_k)_{s_0 \tilde{t}(\tilde{Y})}^{m+n+2D_Y, n+D_Y}(\text{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)),
\end{array}$$

where where the diagonal isomorphisms in (C) and (D) are induced by respective purity isomorphisms.

Diagram (B) commutes by the naturality of the comparison isomorphisms θ_- . Concerning the diagram (C), we have the deformation diagram

$$\begin{array}{ccccc}
\tilde{Y} \times 0 & \xrightarrow{i_0} & \tilde{Y} \times \mathbb{A}^1 & \xleftarrow{i_1} & \tilde{Y} \times 1 \\
s_0 \downarrow & & \downarrow & & \downarrow \iota \\
N_{\tilde{t}} & \xrightarrow{i_0} & \text{Def}(\tilde{t}) & \xrightleftharpoons[\pi_X]{i_1} & \tilde{X} \\
\downarrow & & \downarrow \pi_{\mathbb{A}^1} & & \downarrow \\
0 & \hookrightarrow & \mathbb{A}^1 & \hookleftarrow & 1
\end{array}$$

We see that (C) commutes by applying the Thom isomorphism for the pull-back $\pi_X^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)$ to $\text{Def}(\iota)$ and using the naturality of the Thom isomorphism. One similarly shows that diagram (D) commutes by applying the

comparison isomorphism θ_- to the deformation diagram for the inclusion

$$s_0 \circ \tilde{\iota} : \tilde{Y} \rightarrow \tilde{\nu}_{\tilde{X}} \oplus p_X^* V.$$

The diagrams (A), (B), (C), and (D) fit together to give the diagram (5.5), so it remains to show that diagram (A) commutes.

To see this, we have the canonical isomorphisms (1.7)

$$\det \tilde{\nu}_{\tilde{Y}} \cong p_Y^* \omega_{Y/k}; \quad \det \tilde{\nu}_{\tilde{X}} \cong p_X^* \omega_{X/k}.$$

The exact sequence

$$0 \rightarrow N_{\tilde{\iota}}^{\vee} \rightarrow \iota^* \Omega_{X/k} \rightarrow \Omega_{Y/k} \rightarrow 0$$

gives via the second of these isomorphisms the canonical isomorphism

$$\det(N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \cong p_Y^*(\omega_{Y/k} \otimes \det \bar{V}).$$

Via these isomorphisms, the map ϕ induces the identity map on $p_Y^*(\omega_{Y/k} \otimes \det \bar{V})$, thus by Remark 3.8, we have

$$\phi^* \circ \text{th}(\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \bar{V}) \cup = \text{th}(N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \cup.$$

We have the equality of oriented Thom classes (see Proposition 3.7(2))

$$\text{th}(N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) = p_1^* \text{th}(N_{\tilde{\iota}}) \cup p_2^* \text{th}(\tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V));$$

the identity

$$\text{th}(N_{\tilde{\iota}} \oplus \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \cup = \text{th}(\pi^* \tilde{\iota}^*(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V)) \cup \circ \text{th}(N_{\tilde{\iota}}) \cup$$

follows from this and the functoriality of the oriented Thom class. \square

Putting these two results together yields the identification of pushforward and duality in the case of a closed immersion of smooth projective k -schemes.

Proposition 5.11. *Let $\iota : Y \rightarrow X$ be a codimension r closed immersion of smooth projective k -schemes, let $d_Y = \dim_k Y$ and let $\iota^{\vee} : \Sigma_T^{\infty} Y_+^{\vee} \rightarrow \Sigma_T^{\infty} X_+^{\vee}$ be the dual of $\Sigma_T^{\infty} \iota$. Then the diagram*

$$\begin{array}{ccc} H^m(Y, \mathcal{K}_n^{MW}(\omega_{Y/k})) & \xrightarrow{\iota_*} & H^{m+r}(X, \mathcal{K}_{n+r}^{MW}(\omega_{X/k})) \\ \theta_Y^{\vee} \downarrow & & \downarrow \theta_X^{\vee} \\ H_0(\mathbb{S}_k)^{m+n-2d_Y, n-d_Y}((\Sigma_T^{\infty} Y_+)^{\vee}) & \xrightarrow{\iota_{\vee}^*} & H_0(\mathbb{S}_k)^{m+n-2d_Y, n-d_Y}((\Sigma_T^{\infty} X_+)^{\vee}) \end{array}$$

commutes.

We conclude this section with a result on the compatibility of the pushforward maps ι_* and ι_{\vee}^* with respect to external products. Given k -schemes Y, Y' and vector bundles $W \rightarrow Y, W' \rightarrow Y'$, we write $W \boxplus W'$ for $p_1^* W \oplus p_2^* W'$ and $W \boxtimes W'$ for $p_1^* W \otimes_{\mathcal{O}_{Y \times Y'}} p_2^* W'$.

Suppose we have smooth quasi-projective k -schemes X and X' with locally closed immersions $X \subset \mathbb{P}^d$, $X' \subset \mathbb{P}^{d'}$ and with vector bundles V on X and V' on X' of ranks s and s' , respectively. We have the isomorphism

$$(5.6) \quad \Sigma_T^\infty(X; V)_+^{\tilde{V}} \wedge \Sigma_T^\infty(X'; V')_+^{\tilde{V}} \\ \cong \Sigma^{-d^2-2d-s-d'^2-2d'-s'} \Sigma_T^\infty \text{Th}((\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \boxplus (\tilde{\nu}_{\tilde{X}'} \oplus p_{X'}^* V')).$$

We define the isomorphism

$$H^a(X \times X', \mathcal{K}_b^{MW}(\omega_{X \times X'/k} \otimes \det V \boxtimes \det V')) \\ \xrightarrow{\theta_{X,V}^\vee \wedge \theta_{X',V'}^\vee} H_0(\mathbb{S}_k)^{a+b-2d_X-2d_{X'}, b-d_X-d_{X'}} (\Sigma_T^\infty(X; V)_+^{\tilde{V}} \wedge \Sigma_T^\infty(X'; V')_+^{\tilde{V}})$$

by composing the isomorphism (with $N = d^2 + 2d - d_X + s + d'^2 + 2d' - d_{X'} + s'$)

$$H^a(\tilde{X} \times \tilde{X}', \mathcal{K}_b^{MW}((p_{X \times X'}^*(\omega_{X \times X'/k} \otimes \det V \boxtimes \det V')))) \\ \xrightarrow{\vartheta_{(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \boxplus (\tilde{\nu}_{\tilde{X}'} \oplus p_{X'}^* V')}} H_0(\mathbb{S}_k)^{a+b+2N, b+N} (\text{Th}(\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \boxplus (\tilde{\nu}_{\tilde{X}'} \oplus p_{X'}^* V))$$

with the map on $H_0(\mathbb{S}_k)^{*,*}$ induced by the inverse of the isomorphism (5.6).

Lemma 5.12. *Suppose we have smooth locally closed subschemes $X \subset \mathbb{P}^d$, $X' \subset \mathbb{P}^{d'}$, closed immersions $\iota : Y \rightarrow X$, $\iota' : Y' \rightarrow X'$ in \mathbf{Sm}/k , of codimension r and r' , respectively, and vector bundles V on X and V' on X' . Let $r'' = r + r'$. Then the diagram*

$$\begin{array}{ccc} H^m(Y \times Y', \mathcal{K}_n^{MW}(\omega_{Y \times Y'/k} \otimes \iota^* \det V \boxtimes \iota'^* \det V')) & & \\ \theta_{Y, \iota^* V}^\vee \wedge \theta_{Y', \iota'^* V'}^\vee \downarrow \wr & \searrow (\iota \times \iota')_* & \\ H^{m+r''}(X, \mathcal{K}_{n+r''}^{MW}(\omega_{X \times X'/k} \otimes \det V \boxtimes \det V')) & & \\ \downarrow & & \theta_{X,V}^\vee \wedge \theta_{X',V'}^\vee \downarrow \wr \\ H_0(\mathbb{S}_k)^{m+n-2d_Y, -2d_{Y'}, n-d_Y-d_{Y'}} (\Sigma_T^\infty(Y; \iota^* V)_+^{\tilde{V}} \wedge \Sigma_T^\infty(Y'; \iota'^* V')_+^{\tilde{V}}) & & \\ & \searrow (\iota^{\tilde{V}} \wedge \iota'^{\tilde{V}})^* & \\ & H_0(\mathbb{S}_k)^{m+n-2d_Y, n-d_Y} (\Sigma_T^\infty(X; V)_+^{\tilde{V}} \wedge \Sigma_T^\infty(X'; V')_+^{\tilde{V}}) & \end{array}$$

commutes.

Proof. This follows from Lemma 5.10, noting that the diagram

$$\begin{array}{ccc}
 \Sigma_T^\infty(X; V)_+^{\tilde{\vee}} \wedge \Sigma_T^\infty(X'; V')_+^{\tilde{\vee}} & \xrightarrow[\sim]{(5.6)} & \Sigma^{-N} \Sigma_T^\infty \text{Th}((\tilde{\nu}_{\tilde{X}} \oplus p_X^* V) \boxplus (\tilde{\nu}_{\tilde{X}'} \oplus p_X^* V)) \\
 \downarrow \iota^{\tilde{\vee}} \wedge \iota'^{\tilde{\vee}} & & \downarrow (\iota \times \iota')^{\tilde{\vee}} \\
 \Sigma_T^\infty(Y; \iota^* V)_+^{\tilde{\vee}} \wedge \Sigma_T^\infty(Y'; \iota'^* V')_+^{\tilde{\vee}} & \xrightarrow[\sim]{(5.6)} & \Sigma^{-N} \Sigma_T^\infty \text{Th}((\tilde{\nu}_{\tilde{Y}} \oplus p_Y^* \iota^* V) \boxplus (\tilde{\nu}_{\tilde{Y}'} \oplus p_Y^* \iota'^* V'))
 \end{array}$$

commutes. \square

5.3. Pushforward in Milnor-Witt cohomology: the case of a projection. Next, we define the pushforward map in Milnor-Witt cohomology for a projection $p : \mathbb{P}^N \times X \rightarrow X$, $X \in \mathbf{Sm}/k$.

Recall the affine space bundle $p : \tilde{\mathbb{P}}^d \rightarrow \mathbb{P}^d$ and the rank $d^2 + d$ vector bundle $\tilde{\nu}_{\tilde{\mathbb{P}}^d}$ on $\tilde{\mathbb{P}}^d$. For $X \subset \mathbb{P}^d$ a smooth quasi-projective scheme, we have the pullback $\tilde{X} := p^{-1}(X)$, with map $p_X : \tilde{X} \rightarrow X$ and normal bundle \tilde{N}_X of \tilde{X} in $\tilde{\mathbb{P}}^d$. We have as well the restriction $i_X^* \tilde{\nu}_{\tilde{\mathbb{P}}^d}$ and we have defined

$$\tilde{\nu}_{\tilde{X}} := \tilde{N}_X \oplus i_X^* \tilde{\nu}_{\tilde{\mathbb{P}}^d}.$$

We have the isomorphism

$$\begin{aligned}
 H^a(\mathbb{P}^N \times X, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V)) \\
 \xrightarrow{\theta_{\mathbb{P}^N}^{\vee} \wedge \theta_{X, V}^{\vee}} H_0(\mathbb{S}_k)^{a+b-2N-2d_X, b-N-d_X} (\Sigma_T^\infty \mathbb{P}_+^{N\vee} \wedge \Sigma_T^\infty(X; V)_+^{\tilde{\vee}}).
 \end{aligned}$$

defined in the previous section.

Definition 5.13. Let $X \subset \mathbb{P}^d$ be quasi-projective and let V be a rank $s \geq 0$ vector bundle on X . Let $\pi : \mathbb{P}^N \rightarrow \text{Spec } k$ be the structure morphism. The pushforward map

$$\begin{aligned}
 H^{m+N}(\mathbb{P}^N \times X, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p_2^* \det V)) \\
 \xrightarrow{p_{2*}} H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k} \otimes \det V))
 \end{aligned}$$

is the unique map making the diagram

$$\begin{array}{ccc}
H^{m+N}(\mathbb{P}^N \times X, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V)) & & \\
\downarrow p_{2*} & \searrow \theta_{\mathbb{P}^N}^\vee \wedge \theta_{X,V}^\vee & \\
& \sim & \\
& H_0(\mathbb{S}_k)^{m+n-2d_X, n-d_X}(\Sigma_T^\infty \mathbb{P}_+^{N\vee} \wedge \Sigma_T^\infty(X; V)_+^{\tilde{\vee}}) & \\
& \downarrow (\pi^\vee \wedge \text{Id})^* & \\
H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k} \otimes \det V)) & \searrow \theta_{X,V}^\vee & \\
& \sim & \\
& H_0(\mathbb{S}_k)^{m+n-2d_X, n-d_X}(\Sigma_T^\infty(X; V)_+^{\tilde{\vee}}) &
\end{array}$$

commute.

As it stands, p_{2*} depends on the choice of a locally closed immersion $X \subset \mathbb{P}^d$, but we will eventually identify p_{2*} with Fasel's pushforward map p_{2*}^ϕ , which is independent of any such choice. In what follows, we tacitly assume we have fixed a choice of locally closed immersion $X \subset \mathbb{P}^d$.

Remarks 5.14. (1) Let $j : U \rightarrow X$ be an open immersion and let V be a vector bundle on X . Then the diagram

$$\begin{array}{ccc}
H^{m+N}(\mathbb{P}^N \times X, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p_2^* \det V)) & & \\
\downarrow (\text{Id} \times j)^* & \searrow p_{2*} & \\
& H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k} \otimes \det V)) & \\
& \downarrow j^* & \\
H^{m+N}(\mathbb{P}^N \times U, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times U/k} \otimes p_2^* \det V)) & \searrow p_{2*} & \\
& H^m(U, \mathcal{K}_n^{MW}(\omega_{U/k} \otimes \det V)) &
\end{array}$$

commutes. This follows directly from the definitions and Remark 4.2, noting that the restriction of $\tilde{\nu}_X$ to \tilde{U} is $\tilde{\nu}_U$.

(2) Let $\iota : Y \rightarrow X$ be a codimension r closed immersion in \mathbf{Sm}/k and

let V be a vector bundle on X . Then the diagram

$$\begin{array}{ccc}
H^{m+N}(\mathbb{P}^N \times Y, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times Y/k} \otimes p_2^* i^* \det V)) & & \\
\downarrow (\text{Id} \times \iota)_* & \searrow p_{2*} & \\
& H^m(Y, \mathcal{K}_n^{MW}(\omega_{Y/k} \otimes \iota^* \det V)) & \\
& \downarrow \iota_* & \\
H^{m+d+N}(\mathbb{P}^N \times X, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p_2^* \det V)) & \searrow p_{2*} & \\
& H^{m+d}(X, \mathcal{K}_{n+d}^{MW}(\omega_{X/k} \otimes \det V)) &
\end{array}$$

commutes. This follows from the commutativity of the diagram

$$\begin{array}{ccc}
\Sigma_T^\infty \mathbb{P}_+^{N\vee} \wedge \Sigma_T^\infty(Y; \iota^* V)_+^{\tilde{\vee}} & \xrightarrow{\text{Id} \wedge \iota_V^{\tilde{\vee}}} & \Sigma_T^\infty \mathbb{P}_+^{N\vee} \wedge \Sigma_T^\infty(X; V)_+^{\tilde{\vee}} \\
\downarrow \pi^\vee \wedge \text{Id} & & \downarrow \pi^\vee \wedge \text{Id} \\
\Sigma_T^\infty(Y; \iota^* V)_+^{\tilde{\vee}} & \xrightarrow{\iota_V^{\tilde{\vee}}} & \Sigma_T^\infty(X; V)_+^{\tilde{\vee}},
\end{array}$$

which is a formal consequence of the fact that $\text{SH}(k)$ is a symmetric monoidal category, and Lemma 5.12.

Lemma 5.15. *Suppose that $X \in \mathbf{Sm}/k$ is projective and $V = 0$ is the zero vector bundle. Then*

$$p_{2*} : H^{m+N}(\mathbb{P}^N \times X, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times X/k})) \rightarrow H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k}))$$

is the unique map making the diagram

$$\begin{array}{ccc}
H^{m+N}(\mathbb{P}^N \times X, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}^N \times X/k})) & & \\
\downarrow p_{2*} & \searrow \theta_{\mathbb{P}^N \times X}^\vee & \\
& H_0(\mathbb{S}_k)^{m+n-2d_X, n-d_X}((\Sigma_T^\infty \mathbb{P}^N \times X)_+^\vee) & \\
& \downarrow p_2^{\vee*} & \\
H^m(X, \mathcal{K}_n^{MW}(\omega_{X/k})) & \searrow \theta_X^\vee & \\
& H_0(\mathbb{S}_k)^{m+n-2d_X, n-d_X}(\Sigma_T^\infty X_+^\vee) &
\end{array}$$

commute.

Proof. We have the isomorphism

$$(\Sigma_T^\infty \mathbb{P}^N \times X)_+^\vee \cong (\Sigma_T^\infty \mathbb{P}_+^N)^\vee \wedge \Sigma_T^\infty X_+^\vee$$

via which p_2^\vee transforms to $\pi^\vee \wedge \text{Id}$ and $\theta_{\mathbb{P}^N}^\vee \wedge \theta_X^\vee$ transforms to $\theta_{\mathbb{P}^N \times X}^\vee$. \square

We now turn to the question of comparing our pushforward map with Fasel's, in the case of a projection.

If F is a finitely generated field extension of k , we can find a smooth quasi-projective integral k -scheme X with function field $k(X) = F$, and we define

$$H^a(\text{Spec } F, \mathcal{K}_b^{MW}(\omega_{F/k})) := \text{colim}_U H^a(U, \mathcal{K}_b^{MW}(\omega_{U/k}))$$

where U runs over dense open subschemes of X . We make an analogous definition of $H^a(\mathbb{P}_F^N, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}_F^N/k}))$.

Lemma 5.16. *Let $F \supset k$ be a finitely generated extension field of k , let $a \in \mathbb{P}^N(F)$ be an F -point and let $i_a : \text{Spec } F \rightarrow \mathbb{P}_F^N$ be the corresponding closed immersion. Then the map*

$$i_{a*} : H^0(\text{Spec } F, \mathcal{K}_n^{MW}(\omega_{F/k})) \rightarrow H^N(\mathbb{P}_F^N, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}_F^N/k}))$$

is an isomorphism, independent of the choice of a . Moreover, $p_ \circ i_{a*} = \text{Id}$ on $H^0(\text{Spec } F, \mathcal{K}_n^{MW}(\omega_{F/k}))$.*

Proof. The independence on the choice of a follows from the fact that \mathbb{P}_F^N is \mathbb{A}^1 -connected. To continue the proof, we may take $a = [1 : 0 : \dots : 0]$. In this case, $i_a = \bar{i}_a \times \text{Id}_{\text{Spec } F}$, where $\bar{i}_a : \text{Spec } k \rightarrow \mathbb{P}^N$ is the inclusion corresponding to the k -point $a \in \mathbb{P}^N(k)$. Similarly, $p : \mathbb{P}^N \times \text{Spec } F \rightarrow \text{Spec } F$ is $\pi \times \text{Id}_{\text{Spec } F}$, where $\pi : \mathbb{P}^N \rightarrow \text{Spec } k$ is the structure morphism.

Fasel [13, proof of Theorem 11.7] shows that

$$p_*^\phi : H^N(\mathbb{P}_F^N, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}_F^N/F})) \rightarrow H^0(\text{Spec } F, \mathcal{K}_n^{MW})$$

is an isomorphism. Since $\omega_{F/k} \cong F$ (non-canonically) and $\omega_{\mathbb{P}_F^N/F} \cong \omega_{\mathbb{P}^N/F} \otimes p^* \omega_{F/k}$ (canonically), this shows that

$$p_*^\phi : H^N(\mathbb{P}_F^N, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}_F^N/k})) \rightarrow H^0(\text{Spec } F, \mathcal{K}_n^{MW}(\omega_{F/k}))$$

is an isomorphism as well. By functoriality of pushforward (for Fasel's maps), this implies that

$$i_{a*}^\phi : H^0(\text{Spec } F, \mathcal{K}_n^{MW}(\omega_{F/k})) \rightarrow H^N(\mathbb{P}_F^N, \mathcal{K}_{n+N}^{MW}(\omega_{\mathbb{P}_F^N/k}))$$

is an isomorphism and we conclude by using the identity $i_{a*}^\phi = i_{a*}$ (Proposition 5.3).

It remains to see that $p_* \circ i_{a*} = \text{Id}$ on $H^0(\text{Spec } F, \mathcal{K}_n^{MW}(\omega_{F/k}))$. By Lemma 5.12 and Proposition 5.12, i_{a*} corresponds to the map

$$\begin{aligned} & H_0(\mathbb{S}_k)^{n,n}((\Sigma_T^\infty \text{Spec } F_+)^{\tilde{\vee}}) \\ & \xrightarrow{(\bar{i}_a^\vee \wedge \text{Id})^*} H_0(\mathbb{S}_k)^{n-2d_F, n-d_F}((\Sigma_T^\infty \mathbb{P}_+^N)^\vee \wedge (\Sigma_T^\infty \text{Spec } F_+)^{\tilde{\vee}}), \end{aligned}$$

via the isomorphisms

$$H^N(\mathbb{P}_F^N, \mathcal{K}_{n+N}^{MW}(\omega_{P^N/k})) \xrightarrow{\theta_{\mathbb{P}^N}^\vee \wedge \theta_{\text{Spec } F}^\vee} H_0(\mathbb{S}_k)^{n-2d_F, n-d_F}((\Sigma_T^\infty \mathbb{P}_+^N)^\vee \wedge (\Sigma_T^\infty \text{Spec } F_+)^\vee).$$

and

$$H^0(\text{Spec } F, \mathcal{K}_n^{MW}(\omega_{F/k})) \xrightarrow{\theta_{\text{Spec } F}^\vee} H_0(\mathbb{S}_k)^{n,n}((\Sigma_T^\infty \text{Spec } F_+)^\vee).$$

Thus $p_* \circ i_{a*}$ corresponds to the composition

$$(\pi^\vee \wedge \text{Id})^* \circ (\bar{i}_a^\vee \wedge \text{Id})^* = ([\bar{i}_a^\vee \circ \pi^\vee] \wedge \text{Id})^*$$

which is the identity map, since

$$\bar{i}_a^\vee \circ \pi^\vee = (\pi \circ \bar{i}_a)^\vee = \text{Id}.$$

□

Theorem 5.17. *Let X and Y be smooth quasi-projective k -schemes, let $f : Y \rightarrow X$ be a projective morphism of relative dimension r in \mathbf{Sm}/k , and let V be a vector bundle on X . Choose a locally closed immersion $X \subset \mathbb{P}^d$ and a factorization $f = p \circ i$ for some closed immersion $i : Y \rightarrow \mathbb{P}^N \times X$, $p_2 : \mathbb{P}^N \times X \rightarrow X$ the projection, and define $f_* := p_{2*} \circ i_*$. Then*

$$f_* = f_*^\phi : H^a(Y, \mathcal{K}_b^{MW}(\omega_{Y/k} \otimes f^* \det V)) \rightarrow H^{a-d}(X, \mathcal{K}_{b-d}^{MW}(\omega_{X/k} \otimes \det V)).$$

Proof. We may compose the locally closed immersion $\mathbb{P}^N \times X \subset \mathbb{P}^N \times \mathbb{P}^d$ with the Segre embedding $\mathbb{P}^N \times \mathbb{P}^d \subset \mathbb{P}^M$ to give the locally closed immersion $\mathbb{P}^N \times X \subset \mathbb{P}^M$ needed to define the map i_* . By the functoriality for Fasel's pushforward maps, we have $f_*^\phi = p_{2*}^\phi \circ i_*^\phi$. We have already checked the case of a closed immersion in Proposition 5.3, that is, $i_* = i_*^\phi$, so we may assume $Y = \mathbb{P}^N \times X$ and $f : \mathbb{P}^N \times X \rightarrow X$ is the projection $p := p_2$.

The pushforward map p_*^ϕ is defined to be the map on cohomology induced by the map of Gersten complexes

$$C^*(\mathbb{P}^N \times X, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V)) \xrightarrow{C(p)_*} C^*(X, \mathcal{K}_{b-d}^{MW}(\omega_{X/k} \otimes \det V))[-N]$$

defined by Fasel. In each degree $i \geq N$, the map

$$C^i(\mathbb{P}^N \times X, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V)) \xrightarrow{C(p)_*^i} C^{i-N}(X, \mathcal{K}_{b-d}^{MW}(\omega_{X/k} \otimes \det V))$$

is local over X , that is, for each $x \in X^{(i-N)}$, $y \in (\mathbb{P}^N \times X)^{(i)}$, the component

$$\mathcal{K}_{b-i}^{MW}(k(y))(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V \otimes \det^{-1}(\mathfrak{m}_y/\mathfrak{m}_y^2)) \xrightarrow{C(p)_*^i(y,x)} \mathcal{K}_{b-N-i}^{MW}(k(x))(\omega_{X/k} \otimes \det V \otimes \det^{-1}(\mathfrak{m}_x/\mathfrak{m}_x^2))$$

is zero if $y \notin \mathbb{P}^N \times x \subset \mathbb{P}^N \times X$. Furthermore, if $y \in \mathbb{P}^N \times X$ is a codimension i point and $p(y)$ has codimension $> i - N$, then $C(p)_*^i(y, x) = 0$ for all $x \in X^{(i-N)}$.

We consider the pro-scheme $X_{(i)}$ formed by removing all closed subsets C of X of codimension $> i - N$, that is, we define

$$H^*(X_{(i)}, \mathcal{K}_*^{MW}(\omega_{X_{(i)}/k} \otimes \det V)) := \operatorname{colim}_C H^*(X \setminus C, \mathcal{K}_*^{MW}(\omega_{X \setminus C/k} \otimes \det V)).$$

We may similarly remove all closed subsets $\mathbb{P}^N \times C$ of $\mathbb{P}^N \times X$, $\operatorname{codim}_X C > i$, forming the proscheme $\mathbb{P}^N \times X_{(i)}$ and defining the cohomology group $H^*(\mathbb{P}^N \times X_{(i)}, \mathcal{K}_*^{MW}(\omega_{\mathbb{P}^N \times X_{(i)}/k} \otimes p^* \det V))$ as above. We set $X_{(i)}^{(j)} := X^{(j)} \cap X_{(i)}$ and $\mathbb{P}^N \times X_{(i)}^{(j)} := (\mathbb{P}^N \times X)^{(j)} \cap \mathbb{P}^N \times X_{(i)}$.

We further note that in forming $\mathbb{P}^N \times X_{(i)}$, we have removed all closed subsets of $\mathbb{P}^N \times X$ of codimension $> i$. Thus

$$C^j(\mathbb{P}^N \times X_{(i)}, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X_{(i)}/k} \otimes p^* \det V)) = 0$$

for $j > i$ and

$$\begin{aligned} & C^i(\mathbb{P}^N \times X_{(i)}, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X_{(i)}/k} \otimes p^* \det V)) \\ &= \bigoplus_{y \in \mathbb{P}^N \times X_{(i)}^{(i)}} \mathcal{K}_{b-i}^{MW}(k(y))(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V \otimes \det^{-1}(\mathfrak{m}_y/\mathfrak{m}_y^2)), \end{aligned}$$

giving the surjection

$$\begin{aligned} & \bigoplus_{y \in \mathbb{P}^N \times X_{(i)}^{(i)}} \mathcal{K}_{b-i}^{MW}(k(y))(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V \otimes \det^{-1}(\mathfrak{m}_y/\mathfrak{m}_y^2)) \\ & \rightarrow H^i(\mathbb{P}^N \times X_{(i)}, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X_{(i)}/k} \otimes p^* \det V)). \end{aligned}$$

Since we have removed all closed subsets of X of codimension $> i - N$, the codimension $i - N$ points of X are the closed points of $X_{(i)}$. Fix some $x \in X^{(i-N)}$, giving us the cartesian diagram

$$\begin{array}{ccccc} \mathbb{P}^N \times x & \xrightarrow{\operatorname{Id} \times i_x} & \mathbb{P}^N \times X_{(i)} & \xrightarrow{\operatorname{Id} \times j} & \mathbb{P}^N \times X \\ p_x \downarrow & & \downarrow p & & \downarrow p \\ x & \xrightarrow{i_x} & X_{(i)} & \xrightarrow{j} & X \end{array}$$

with j a (pro)-open immersion and i_x a (pro)-closed immersion. By Remark 5.14, we have

$$j^* p_* = p_*(\operatorname{Id} \times j)^*; \quad p_*(\operatorname{Id} \times i_x)_* = i_{x*} p_{x*}.$$

Looking at the Gersten complexes, we see that

$$j^* : H^{i-N}(X, \mathcal{K}_{b-N}^{MW}(\omega_{X/k} \otimes \det V)) \rightarrow H^{i-N}(X_{(i)}, \mathcal{K}_{b-N}^{MW}(\omega_{X/k} \otimes \det V))$$

is injective and

$$\begin{aligned} & \oplus_{x \in X^{(i-N)}} H^N(\mathbb{P}^N \times x, \mathcal{K}_{b+N-i}^{MW}(\omega_{\mathbb{P}^N \times x/k} \otimes p_x^* i_x^* \det V)) \\ & \xrightarrow{\sum_{x \in X^{(i-N)}} (\text{Id} \times i_x)_*} H^i(\mathbb{P}^N \times X_{(i)}, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V)) \end{aligned}$$

is surjective. As $i_{x*} = i_{x*}^\phi$, we need only show that $p_{x*} = p_{x*}^\phi$. Choosing a k -point a of \mathbb{P}^N , Lemma 5.16 tells us that p_{x*} and p_{x*}^ϕ are both inverse to the isomorphism

$$\begin{aligned} & H^0(x, \mathcal{K}_{b-i}^{MW}(\omega_{x/k} \otimes i_x^* \det V)) \\ & \xrightarrow{i_{a*}^\phi} H^N(\mathbb{P}^N \times x, \mathcal{K}_{b+N-i}^{MW}(\omega_{\mathbb{P}^N \times x/k} \otimes p_x^* i_x^* \det V)), \end{aligned}$$

completing the proof. \square

Corollary 5.18. 1. Let $\iota : Y \rightarrow X$ be a codimension r closed immersion among smooth quasi-projective k -schemes and let V be a vector bundle on X . Then the pushforward map

$$\iota_* : H^a(Y, \mathcal{K}_b^{MW}(\omega_{Y/k} \otimes \iota^* \det V)) \rightarrow H^{a+r}(X, \mathcal{K}_{b+r}^{MW}(\omega_{X/k} \otimes \det V))$$

is independent of the choice of locally closed immersion $X \subset \mathbb{P}^d$.

2. Let X be a smooth quasi-projective k -scheme with a vector bundle V , $p : \mathbb{P}^N \times X \rightarrow X$ the projection. Then the pushforward map

$$\begin{aligned} & H^a(\mathbb{P}^N \times X, \mathcal{K}_b^{MW}(\omega_{\mathbb{P}^N \times X/k} \otimes p^* \det V)) \\ & \xrightarrow{p_*} H^{a-N}(X, \mathcal{K}_{b-N}^{MW}(\omega_{X/k} \otimes \det V)) \end{aligned}$$

is independent of the choice of locally closed immersion $X \subset \mathbb{P}^d$.

3. Let $f : Y \rightarrow X$ be a projective morphism between smooth quasi-projective k -schemes, of relative dimension d , and let V be a vector bundle on X . Factor f as $p \circ i$ with $i : Y \rightarrow \mathbb{P}^N \times X$ a closed immersion and $p : \mathbb{P}^N \times X \rightarrow X$ the projection and define $f_* := p_* \circ i_*$. Then the map

$$f_* : H^a(Y, \mathcal{K}_b^{MW}(\omega_{Y/k} \otimes f^* \det V)) \rightarrow H^{a-d}(X, \mathcal{K}_{b-d}^{MW}(\omega_{X/k} \otimes \det V))$$

is independent of the choice of factorization. Moreover, if $g : X \rightarrow W$ is a projective morphism between smooth quasi-projective k -schemes, of relative dimension e , and V is a vector bundle on W , then $(g \circ f)_* = g_* \circ f_*$ as maps

$$H^a(Y, \mathcal{K}_b^{MW}(\omega_{Y/k} \otimes (gf)^* \det V)) \rightarrow H^{a-d-e}(W, \mathcal{K}_{b-d-e}^{MW}(\omega_{W/k} \otimes \det V))$$

Proof. These all follow from Theorem 5.17 and the functoriality of Fasel's pushforward maps. \square

Corollary 5.19. *Let $f : Y \rightarrow X$ be a morphism of smooth projective k -schemes, of relative dimension r . Then the diagram*

$$\begin{array}{ccc} H^a(Y, \mathcal{K}_b^{MW}(\omega_{Y/k})) & \xrightarrow{f_*} & H^{a-r}(X, \mathcal{K}_{b-r}^{MW}(\omega_{X/k})) \\ \theta_Y^\vee \downarrow \wr & & \wr \downarrow \theta_X^\vee \\ H_0(\mathbb{S}_k)^{a+b-2d_Y, b-d_Y}(\Sigma_T^\infty Y_+^\vee) & \xrightarrow{f^{\vee*}} & H_0(\mathbb{S}_k)^{a+b-2d_Y, b-d_Y}(\Sigma_T^\infty X_+^\vee) \end{array}$$

commutes.

Proof. Since X and Y are projective, f is a projective morphism. Factor f as a closed immersion $i : Y \rightarrow \mathbb{P}^N \times X$ followed by the projection $p : \mathbb{P}^N \times X \rightarrow X$. The assertion then follows from Proposition 5.11, Lemma 5.15 and the identity $f^\vee = i^\vee \circ p^\vee$. \square

6. THE PROOF OF THEOREM 1

We retain the notation of §1. Let X be a smooth integral projective k -scheme. We fix an closed immersion $X \hookrightarrow \mathbb{P}^d$ and let $N = d^2 + 2d$. By Lemma 1.5, the composition

$$\begin{aligned} \text{GW}(k) = H_0(\mathbb{S}_k)^{2N, N}(T^N) & \xrightarrow{\Sigma_T^N \pi_{\tilde{X}}^*} H_0(\mathbb{S}_k)^{2N, N}(\Sigma_T^N X_+) \\ & \xrightarrow{\text{Th}(\beta_{X \subset \mathbb{P}^d})^*} H_0(\mathbb{S}_k)^{2N, N}(\text{Th}(\tilde{\nu}_{\tilde{X}})) \\ & \xrightarrow{\eta_{X \subset \mathbb{P}^d}^*} H_0(\mathbb{S}_k)^{2N, N}(T^N) = \text{GW}(k) \end{aligned}$$

is multiplication by $\chi^{\text{cat}}(X)$.

We have the canonical isomorphism $N_{\tilde{\Delta}_X} \cong p_X^* T_X$. Via (1.7) we have the canonical isomorphism

$$\det(N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}}) \cong \det T_X \otimes p_X^*(\omega_{X/k}) \cong O_{\tilde{X}},$$

and via this isomorphism, the isomorphism (1.9) $\psi : O_{\tilde{X}}^N \rightarrow N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}}$ induces the identity isomorphism on $O_{\tilde{X}}$ upon applying the operation \det . Letting $V = N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}}$, Corollary 4.4 gives us the commutative diagram

$$\begin{array}{ccc} H^0(\tilde{X}, \mathcal{K}_0^{MW}) & \xrightarrow{\vartheta_V} & H_0(\mathbb{S}_k)^{2N, N}(\text{Th}(V)) \\ \alpha_{\tilde{X}} \downarrow & & \downarrow \text{Th}(\psi)^* \\ H_0(\mathbb{S}_k)^{0, 0}(\tilde{X}) & \xrightarrow{\Sigma_T^N = \vartheta_{O_{\tilde{X}}^N}} & H_0(\mathbb{S}_k)^{2N, N}(\Sigma_T^N \tilde{X}_+) \end{array}$$

where $\alpha_{\tilde{X}}$ is the canonical comparison isomorphism (4.1). Similarly, we have the commutative diagrams

$$\begin{array}{ccc} H^0(\mathrm{Spec} k, \mathcal{K}_0^{MW}) & \xrightarrow{\vartheta_{k^N}} & H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}(k^N)) \\ \alpha_{\mathrm{Spec} k} \downarrow & & \parallel \\ H_0(\mathbb{S}_k)^{0,0}(\mathrm{Spec} k) & \xrightarrow{\Sigma_T^N} & H_0(\mathbb{S}_k)^{2N,N}(T^N) \end{array}$$

Together with the naturality of the various comparison isomorphisms, these give us the commutative diagram

(6.1)

$$\begin{array}{ccccc} H^0(\mathrm{Spec} k, \mathcal{K}_0^{MW}) & \xrightarrow{\pi_{\tilde{X}}^*} & H^0(\tilde{X}, \mathcal{K}_0^{MW}) & & \\ \vartheta_{k^N} \downarrow & & \downarrow \vartheta_{O_{\tilde{X}}^N} & \searrow \vartheta_V & \\ H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}(k^N)) & \xrightarrow{\Sigma_T^N \pi_{\tilde{X}}^*} & H_0(\mathbb{S}_k)^{2N,N}(\Sigma_T^N X_+) & \xleftarrow[\mathrm{Th}(\psi)^*]{\sim} & H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}(V)) \end{array}$$

We recall that $\beta_{X \subset \mathbb{P}^d}$ is the composition of the inclusion $i_2 : \tilde{\nu}_{\tilde{X}} \rightarrow V = N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}}$ as the second summand followed by the isomorphism $\psi^{-1} : V \rightarrow O_{\tilde{X}}^N$.

We claim the diagram

$$(6.2) \quad \begin{array}{ccc} H^0(\tilde{X}, \mathcal{K}_0^{MW}) & \xrightarrow{e(p_X^* T_X) \cup (-)} & H^{d_X}(\tilde{X}, \mathcal{K}_{d_X}^{MW}(\omega_{X/k})) \\ \vartheta_V \downarrow & & \downarrow \vartheta_{\tilde{\nu}_{\tilde{X}}} \\ H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}(V)) & \xrightarrow{\mathrm{Th}_{\tilde{X}}(i_2)^*} & H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}_{\tilde{X}}(\tilde{\nu}_{\tilde{X}})) \end{array}$$

commutes. To see this, we recall that $\vartheta_V = \theta_V \circ \mathrm{th}(V) \cup$, where

$$\theta_V : H_{s_0(\tilde{X})}^{m+N}(V, \mathcal{K}_{n+N}^{MW}) \rightarrow H_0^{m+n+2N, n+N}(\mathbb{S}_k)(\mathrm{Th}(V))$$

is the comparison isomorphism, and

$$\mathrm{th}(V) \cup : H^m(\tilde{X}, \mathcal{K}_n^{MW}) \rightarrow H_{s_0(\tilde{X})}^{m+N}(V, \mathcal{K}_{n+N}^{MW})$$

is the map $\mathrm{th}(V) \cup (\alpha) := \mathrm{th}(V) \cup \pi_V^* \alpha$, with $\pi_V : V \rightarrow \tilde{X}$ the projection. The map $\vartheta_{\tilde{\nu}_{\tilde{X}}}$ has a similar factorization as $\theta_{\tilde{\nu}_{\tilde{X}}} \circ \mathrm{th}(\tilde{\nu}_{\tilde{X}}) \cup$. Since the comparison isomorphisms $\theta_{(-)}$ are natural transformations, we need to show that the

diagram

$$\begin{array}{ccc}
H^0(\tilde{X}, \mathcal{K}_0^{MW}) & \xrightarrow{e(p_X^* T_X) \cup (-)} & H^{d_X}(\tilde{X}, \mathcal{K}_{d_X}^{MW}(p_X^* \omega_{X/k})) \\
\text{th}(V) \cup \downarrow & & \downarrow \text{th}(\tilde{\nu}_{\tilde{X}}) \cup \\
H_{s_0(\tilde{X})}^N(V, \mathcal{K}_N^{MW}) & \xrightarrow{i_2^*} & H_{s_0(\tilde{X})}^N(\tilde{\nu}_{\tilde{X}}, \mathcal{K}_N^{MW})
\end{array}$$

commutes. Since $V = N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}}$, we have $\text{th}(V) = p_1^*(\text{th}(N_{\tilde{\Delta}_X})) \cup p_2^*(\text{th}(\tilde{\nu}_{\tilde{X}}))$. We have the isomorphism (1.8), $N_{\tilde{\Delta}_X} \cong p_X^* T_X$, inducing an isomorphism on the determinant bundles. Identifying the corresponding twisted Milnor-Witt sheafs using this latter isomorphism we have $e(p_X^* T_X) = e(N_{\tilde{\Delta}_X})$. Thus, for $\alpha \in H^m(\tilde{X}, \mathcal{K}_n^{MW})$, we have

$$\begin{aligned}
i_2^*(\text{th}(V) \cup (\alpha)) &= i_2^*(\text{th}(V) \cup \pi_V^*(\alpha)) \\
&= i_2^*(p_1^*(\text{th}(N_{\tilde{\Delta}_X})) \cup p_2^*(\text{th}(\tilde{\nu}_{\tilde{X}})) \cup \pi_{\tilde{\nu}_{\tilde{X}}}^*(\alpha)) \\
&= i_2^*(p_1^*(\text{th}(N_{\tilde{\Delta}_X})) \cup \text{th}(\tilde{\nu}_{\tilde{X}}) \cup \pi_{\tilde{\nu}_{\tilde{X}}}^*(\alpha)) \\
&\stackrel{(a)}{=} \pi_{\tilde{\nu}_{\tilde{X}}}^*(s_0^* \text{th}(N_{\tilde{\Delta}_X})) \cup \text{th}(\tilde{\nu}_{\tilde{X}}) \cup \pi_{\tilde{\nu}_{\tilde{X}}}^*(\alpha) \\
&= \pi_{\tilde{\nu}_{\tilde{X}}}^*(e(N_{\tilde{\Delta}_X})) \cup \text{th}(\tilde{\nu}_{\tilde{X}}) \cup \pi_{\tilde{\nu}_{\tilde{X}}}^*(\alpha) \\
&\stackrel{(b)}{=} \text{th}(\tilde{\nu}_{\tilde{X}}) \cup \pi_{\tilde{\nu}_{\tilde{X}}}^*(e(N_{\tilde{\Delta}_X}) \cup \alpha) \\
&= \text{th}(\tilde{\nu}_{\tilde{X}}) \cup (e(p_X^* T_X) \cup \alpha).
\end{aligned}$$

Except for (a), and (b) these equalities are evident, and (a) follows from the commutative square

$$\begin{array}{ccc}
\tilde{\nu}_{\tilde{X}} & \xrightarrow{i_2} & N_{\tilde{\Delta}_X} \oplus \tilde{\nu}_{\tilde{X}} \\
\pi_{\tilde{\nu}_{\tilde{X}}} \downarrow & & \downarrow p_1 \\
\tilde{X} & \xrightarrow{s_0} & N_{\tilde{\Delta}_X},
\end{array}$$

which shows that $i_2^* p_1^* = \pi_{\tilde{\nu}_{\tilde{X}}}^* s_0^*$. The equality (b) follows from Proposition 3.9. Thus (6.2) commutes, as claimed.

Finally, we claim that the diagram (6.3)

$$\begin{array}{ccc}
H^{d_X}(\tilde{X}, \mathcal{K}_{d_X}^{MW}(p_X^* \omega_{X/k})) & \xleftarrow{p_X^*} H^{d_X}(X, \mathcal{K}_{d_X}^{MW}(\omega_{X/k})) & \xrightarrow{\pi_{X*}} H^0(\text{Spec } k, \mathcal{K}_0^{MW}) \\
\vartheta_{\tilde{\nu}_{\tilde{X}}} \downarrow & & \downarrow \vartheta_{k^N} \\
H_0(\mathbb{S}_k)^{2N, N}(\text{Th}_{\tilde{X}}(\tilde{\nu}_{\tilde{X}})) & \xrightarrow{\eta_{X \subset \mathbb{P}^d}^*} & H_0(\mathbb{S}_k)^{2N, N}(\text{Th}(k^N))
\end{array}$$

commutes. Indeed, the desuspended stabilization of $\eta_{X \subset \mathbb{P}^d}$ is π_X^\vee , and so we have the commutative diagram

$$\begin{array}{ccc} H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}_{\tilde{X}}(\tilde{\nu}_{\tilde{X}})) & \xrightarrow{\eta_{X \subset \mathbb{P}^d}^*} & H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}(k^N)) \\ \Sigma_T^N \downarrow & & \downarrow \Sigma_T^N \\ H_0(\mathbb{S}_k)^{0,0}(\Sigma_T^\infty X_+^\vee) & \xrightarrow{\pi_X^{\vee*}} & H_0(\mathbb{S}_k)^{0,0}(\Sigma_T^\infty \mathrm{Spec} k_+^\vee) \end{array}$$

where the Σ_T^N denote the respective stabilization isomorphisms. In addition, we have

$$\theta_X^\vee = \Sigma_T^N \circ \vartheta_{\tilde{\nu}_{\tilde{X}}} \circ p_X^*$$

and

$$\theta_{\mathrm{Spec} k}^\vee = \Sigma_T^N \circ \vartheta_{k^N}$$

so the commutativity of (6.3) follows from Corollary 5.19 applied to the morphism $p_X : X \rightarrow \mathrm{Spec} k$.

Putting together the commutative diagrams (6.1), (6.2) and (6.3) and noting that $\mathrm{Th}(i_2^*) \circ \mathrm{Th}(\psi^{-1})^* = \mathrm{Th}_{\tilde{X}}(\beta_{X \subset \mathbb{P}^d})^*$ gives us the commutative diagram

$$\begin{array}{ccc} H^0(k, \mathcal{K}_0^{MW}) & \xrightarrow[\sim]{\vartheta_{k^N}} & H_0(\mathbb{S}_k)^{2N,N}(T^N) \\ \pi_X^* \downarrow & & \downarrow \Sigma_T^N \pi_{\tilde{X}}^* \\ H^0(X, \mathcal{K}_0^{MW}) & \xrightarrow[\sim]{\vartheta_{O_X^N}} & H_0(\mathbb{S}_k)^{2N,N}(\Sigma_T^N \tilde{X}_+) \\ \downarrow e(T_X) \cup (-) & & \downarrow \mathrm{Th}_{\tilde{X}}(\beta_{X \subset \mathbb{P}^d})^* \\ \times \chi^{CW}(X) \quad H^{d_X}(X, \mathcal{K}_{d_X}^{MW}(\omega_{X/k})) & \xrightarrow[\sim]{\vartheta_{\tilde{\nu}_{\tilde{X}}}} & H_0(\mathbb{S}_k)^{2N,N}(\mathrm{Th}_{\tilde{X}}(\tilde{\nu}_{\tilde{X}})) \quad \times \chi^{cat}(X) \\ \pi_{X*} \downarrow & & \downarrow \eta_{X \subset \mathbb{P}^d}^* \\ H^0(k, \mathcal{K}_0^{MW}) & \xrightarrow[\sim]{\vartheta_{k^N}} & H_0(\mathbb{S}_k)^{2N,N}(T^N), \end{array}$$

which shows that $\chi^{cat}(X) = \chi^{CW}(X)$ in $\mathrm{GW}(k)$, as desired.

7. VARIETIES OF ODD DIMENSION

From now on, we write $\chi(X)$ for $\chi^{cat}(X) = \chi^{CW}(X)$. We have already seen that for X a smooth projective k -scheme of odd dimension, there is an integer m such that $\chi(X) - m \cdot h$ is a 2-torsion element of $\mathrm{GW}(k)$. We will use Theorem 1 to improve this.

Theorem 7.1. *Let Y be an integral smooth projective k -scheme of odd dimension over k .*

1. *The Euler characteristic $\chi(Y)$ has even rank.*
2. *If $\mathrm{rank}(\chi(Y)) = 2m$, then $\chi(Y) = m \cdot h$.*

We have already proven (1) in Proposition 1.12, but the proof below does not use any realization functors and relies only on properties of the Euler class and quadratic forms. The proof is based on the following lemma.

Lemma 7.2. *Let $\pi : V \rightarrow Y$ be a vector bundle of odd rank r over some $Y \in \mathbf{Sm}/k$. Then for all $u \in k^\times$,*

$$e(V) = \langle u \rangle \cdot e(V)$$

in $H^r(Y, \mathcal{K}_r^{MW}(\det^{-1} V))$.

Proof. Let $\phi_u : V \rightarrow V$ be the map multiplication by u . The naturality of the Thom class says that

$$(\phi_u, \det^{-1} \phi_u)^*(\mathrm{th}(V)) = \mathrm{th}(V)$$

Since $\det^{-1} \phi_u : \det^{-1} V \rightarrow \det^{-1} V$ is multiplication by u^{-r} , $\det^{-1} \phi_u^*$ is multiplication by $\langle u \rangle^r = \langle u \rangle$ on $H^r(V, \mathcal{K}_r^{MW}(\pi^* \det^{-1} V))$. Since $\phi_u^* \circ \pi^* = \pi^*$, we have the pullback map

$$\phi_u^* : H^r(V, \mathcal{K}_r^{MW}(\pi^* \det^{-1} V)) \rightarrow H^r(V, \mathcal{K}_r^{MW}(\pi^* \det^{-1} V))$$

with $\det^{-1} \phi_u \circ \phi_u^* = (\phi_u, \det^{-1} \phi_u)^*$, and thus

$$\phi_u^*(\mathrm{th}(V)) = \langle u \rangle \cdot \mathrm{th}(V).$$

Since $\phi_u \circ s_0 = s_0$, we have

$$\begin{aligned} e(V) &= s_0^*(\mathrm{th}(V)) \\ &= s_0^*(\phi_u^*(\mathrm{th}(V))) \\ &= s_0^*(\langle u \rangle \cdot \mathrm{th}(V)) \\ &= \langle u \rangle \cdot e(V). \end{aligned}$$

□

Proof of Theorem 7.1. Suppose Y is integral of odd dimension d over k . Applying Lemma 7.2, we have

$$e(T_Y) = \langle u \rangle \cdot e(T_Y) \in H^d(Y, \mathcal{K}_d^{MW}(\omega_{Y/k}))$$

for all $u \in k^\times$; pushing forward to $\mathrm{Spec} k$ gives

$$\chi(Y) = \langle u \rangle \cdot \chi(Y)$$

for all $u \in k^\times$. More generally, if $k \subset F$ is an extension of fields, let $Y_F \in \mathbf{Sm}/F$ denote the base-extension $Y \times_k F$ of Y . Then

$$(7.1) \quad \chi(Y_F) = \langle u \rangle \cdot \chi(Y_F) \in \mathrm{GW}(F)$$

for all $u \in F^\times$.

To prove (1), we may detect the rank n of $\chi(Y)$ by taking the base-extension $k \subset \bar{k}$, \bar{k} the algebraic closure of k . As the map

$$\mathrm{GW}(\bar{k}) \rightarrow \mathrm{GW}(\bar{k}(t))$$

is injective, split by the rank homomorphism $\mathrm{rank} : \mathrm{GW}(\bar{k}(t)) \rightarrow \mathbb{Z}$, it suffices to show that $\chi(Y_{\bar{k}(t)})$ has even rank.

The identity (7.1) gives

$$(1 - \langle t \rangle) \cdot \chi(Y_{\bar{k}(t)}) = 0$$

in $\mathrm{GW}(\bar{k}(t))$. The augmentation ideal $I_{\bar{k}(t)}$ satisfies $I_{\bar{k}(t)}^2 = 0$ and so $I_{\bar{k}(t)} \cong I_{\bar{k}(t)}/I_{\bar{k}(t)}^2 \cong \bar{k}(t)^\times / (\bar{k}(t)^\times)^2$, with $u \in \bar{k}(t)^\times$ mapping to $1 - \langle u \rangle \in I_{\bar{k}(t)}$. Since $\chi(Y_{\bar{k}(t)})$ is the base-extension of $\chi(Y_{\bar{k}}) = n \cdot 1$, we have $\chi(Y_{\bar{k}(t)}) = n \cdot 1$ and thus

$$0 = (1 - \langle t \rangle) \cdot n \cdot 1 = 1 - \langle t^n \rangle,$$

which implies that $t^n = t$ in $\bar{k}(t)^\times / (\bar{k}(t)^\times)^2$, and thus n is even.

For (2), $\chi(Y)$ has rank $n = 2m$ by (1). Consider $\chi(Y) - m \cdot h$. To show this is zero in $\mathrm{GW}(k)$, we may take the base-change to $\mathrm{GW}(k(t))$, so it suffices to show that $\chi(Y_{k(t)}) - m \cdot h = 0$ in $\mathrm{GW}(k(t))$. By (7.1), we have as above

$$\chi(Y_{k(t)}) - m \cdot h = \langle t \rangle \cdot (\chi(Y_{k(t)}) - m \cdot h)$$

in $\mathrm{GW}(k(t)) = K_0^{MW}(k(t))$, since $\langle t \rangle \cdot h = h$.

Let \mathcal{O} be the local ring $k[t]_{(t)}$ with parameter t , giving us the boundary map

$$\partial_t : K_0^{MW}(k(t)) \rightarrow K_{-1}^{MW}(k).$$

For $x \in K_0^{MW}(k) = \mathrm{GW}(k)$ with base-extension $x_{k(t)} \in K_0^{MW}(k(t))$,

$$\partial_t(\langle t \rangle \cdot x_{k(t)}) = \eta \cdot x,$$

which via the isomorphism $K_{-1}^{MW}(k) \cong W(k)$, is the image of x under the canonical map $\mathrm{GW}(k) \rightarrow W(k)$. Similarly, as $x_{k(t)}$ is the restriction of $x_{\mathcal{O}} \in K_0^{MW}(\mathcal{O})$, we have $\partial_t(x_{k(t)}) = 0$. This gives

$$0 = \partial_t(\chi(Y_{k(t)}) - m \cdot h) = \partial_t(\langle t \rangle \cdot (\chi(Y_{k(t)}) - m \cdot h)) = \eta \cdot (\chi(Y_{k(t)}) - m \cdot h).$$

But since $\mathrm{rank}(\chi(Y_{k(t)}) - m \cdot h) = 0$, $\chi(Y_{k(t)}) - m \cdot h$ is in the augmentation ideal $I_{k(t)}$. As the restriction of $\mathrm{GW}(F) \rightarrow W(F)$ to I_F is an isomorphism of I_F with the augmentation ideal of $W(F)$, it follows that $\chi(Y_{k(t)}) - m \cdot h = 0$ in $\mathrm{GW}(k(t))$ and hence

$$\chi(Y) = m \cdot h$$

in $\mathrm{GW}(k)$. □

There is a similar consequence for the Euler classes of odd rank bundles.

Proposition 7.3. *Let V be a vector bundle of odd rank r on $X \in \mathbf{Sm}/k$. Then the Euler class $e(V) \in H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$ satisfies*

$$\eta \cdot e(V) = 0$$

in $H^r(X, \mathcal{K}_{r-1}^{MW}(\det^{-1} V))$.

Proof. The proof is essentially the same as the proof of Theorem 7.1. It suffices to show that $\eta \cdot e(V_{k(t)}) = 0$ in $H^r(X_{k(t)}, \mathcal{K}_{r-1}^{MW}(\det^{-1} V_{k(t)}))$. As above, we have

$$e(V_{k(t)}) = \langle t \rangle \cdot e(V_{k(t)})$$

which gives

$$0 = \partial_t(e(V_{k(t)})) = \partial_t(\langle t \rangle \cdot e(V_{k(t)})) = \eta \cdot e(V_{k(t)}).$$

□

8. LINE BUNDLES

8.1. Euler classes of line bundles, cocycles and sections. The group of isomorphism classes of line bundles on a scheme X , $\text{Pic}(X)$, is classified by taking the corresponding cohomology class $[L] \in H^1(X, \mathcal{O}_X^*)$; via the isomorphism $\mathcal{O}_X^* \cong \mathcal{K}_1^M$, we may view $[L]$ as living in $H^1(X, \mathcal{K}_1^M)$. If X is smooth, taking the divisor of a rational section of a line bundle defines an isomorphism of $\text{Pic}(X)$ with the group of divisors modulo linear equivalence, $\text{CH}^1(X)$. One can also view the isomorphism $H^1(X, \mathcal{K}_1^M) \cong \text{CH}^1(X)$ as arising from the cohomology sequence associated to the exact sheaf sequence

$$1 \rightarrow \mathcal{K}_1^M \rightarrow i_{\eta*} K_1^M(k(X)) \xrightarrow{\partial} \oplus_{x \in X(1)} i_{x*} K_0^M(k(x)) \rightarrow 0,$$

that is, the Gersten sequence for \mathcal{K}_1^M , noting that $\oplus_{x \in X(1)} i_{x*} K_0^M(k(x))$ is the group of divisors on X , $K_1^M(k(X)) = k(X)^\times$ and ∂ is the map sending a rational function to its divisor.

The Euler class $e(L)$ of a line bundle L does not live in $H^1(X, \mathcal{K}_1^{MW})$ but rather in the cohomology of the twisted sheaf, $H^1(X, \mathcal{K}_1^{MW}(L^{-1}))$, a group which itself depends on the given line bundle L . Thus, the Euler class does not classify line bundles, but gives instead a further invariant of a particular line bundle. In spite of this, the situation is similar to the one discussed above, in that there is a cocycle-type construction of $e(L) \in H^1(X, \mathcal{K}_1^{MW}(L^{-1}))$.

We recall the definition of twisted sheaf $\mathcal{K}_*^{MW}(L)_X$ on a smooth k -scheme X with line bundle L as

$$\mathcal{K}_*^{MW}(L)_X = \mathcal{K}_{*X}^{MW} \otimes_{\mathbb{Z}[\mathbb{G}_m]} \mathbb{Z}[L^\times].$$

Suppose that we have a trivialization of L for some Zariski open cover $\{U_\alpha\}$ of X via isomorphisms

$$\phi_\alpha : L|_{U_\alpha} \rightarrow \mathcal{O}_{U_\alpha}$$

and let $\xi_{\alpha\beta} \in \mathcal{O}_X^\times(U_\alpha \cap U_\beta)$ be the transition function

$$\phi_\alpha \circ \phi_\beta^{-1} = \times \xi_{\alpha\beta}$$

on $\mathcal{O}_{U_\alpha \cap U_\beta}$. The section $\lambda_\alpha := \phi_\alpha^{-1}(1)$ of $L|_{U_\alpha}$ gives a generator λ_α of $\mathbb{Z}[L^\times]_{|U_\alpha}$ as a free $\mathbb{Z}[\mathbb{G}_m]_{|U_\alpha}$ -module and thus

$$\mathcal{K}_n^{MW}(L)_{U_\alpha} = \mathcal{K}_{nU_\alpha}^{MW} \otimes \lambda_\alpha$$

as a $\mathcal{K}_{0U_\alpha}^{MW}$ -module.

The automorphism $\mathcal{K}_*^{MW}(\phi_\alpha) \circ \mathcal{K}_*^{MW}(\phi_\beta)^{-1}$ of $\mathcal{K}_{*U_\alpha \cap U_\beta}^{MW}$ is given by multiplication by $\langle \xi_{\alpha\beta} \rangle$, which gives us the usual description of a section s of $\mathcal{K}_n^{MW}(L)_X$ over some U as a collection $\{s_\alpha \in \mathcal{K}_{nX}^{MW}(U \cap U_\alpha)\}_\alpha$ satisfying

$$s_\alpha = \langle \xi_{\alpha\beta} \rangle s_\beta \text{ in } \mathcal{K}_{nX}^{MW}(U \cap U_\alpha \cap U_\beta).$$

Since

$$1 \otimes \lambda_\beta = 1 \otimes \xi_{\alpha\beta} \lambda_\alpha = \langle \xi_{\alpha\beta} \rangle \otimes \lambda_\alpha$$

over $U_\alpha \cap U_\beta$, given local sections s_α as above, we have

$$s_\alpha \otimes \lambda_\alpha = s_\beta \langle \xi_{\alpha\beta} \rangle \otimes \lambda_\alpha = s_\beta \otimes \lambda_\beta$$

over $U \cap U_\alpha \cap U_\beta$, giving the unique section s of $\mathcal{K}_n^{MW}(L)_X$ over U with $s|_{U \cap U_\alpha} = s_\alpha$.

Similarly, we have the usual description of Čech cohomology, for instance, a Čech 1-cocycle for $\mathcal{K}_n^{MW}(L)$ for the cover $\mathcal{U} = \{U_\alpha\}_\alpha$ consists of elements $s_{\alpha\beta} \in \mathcal{K}_n^{MW}(U_\alpha \cap U_\beta)$ such that

$$s_{\alpha\gamma} = s_{\alpha\beta} + \langle \xi_{\alpha\beta} \rangle s_{\beta\gamma}$$

after restriction to $\mathcal{K}_n^{MW}(U_\alpha \cap U_\beta \cap U_\gamma)$.

Remark 8.1. Let L, M be line bundles on X , trivialized by the same open cover $\{U_\alpha\}_\alpha$, with respective local generating sections $\lambda_\alpha, \mu_\alpha$, and with respective transition functions $\xi_{\alpha\beta}^L, \xi_{\alpha\beta}^M$. For $U \subset X$ over, we have the isomorphism $\mathrm{GW}(U, L) \rightarrow \mathrm{GW}(U, L \otimes M^{\otimes 2})$ sending a symmetric isomorphism $\phi : V \rightarrow \mathrm{Hom}(V, L)$ to the composition

$$V \otimes M \xrightarrow{\phi \otimes \mathrm{Id}} \mathrm{Hom}(V, L) \otimes M \cong \mathrm{Hom}(V \otimes M, L \otimes M^{\otimes 2}).$$

This induces the isomorphism of sheaves $\psi_M : \mathcal{GW}(L)_X \rightarrow \mathcal{GW}(L \otimes M^{\otimes 2})_X$ and thereby the isomorphism

$$\begin{aligned} \mathcal{K}_*^{MW}(L) &= \mathcal{K}_*^{MW} \otimes_{\mathcal{GW}} \mathcal{GW}(L) \\ &\xrightarrow{\mathrm{Id} \otimes \psi_M} \mathcal{K}_*^{MW} \otimes_{\mathcal{GW}} \mathcal{GW}(L \otimes M^{\otimes 2}) = \mathcal{K}_*^{MW}(L \otimes M^{\otimes 2}). \end{aligned}$$

In terms of local trivializations, this is the map

$$\mathcal{K}_n^{MW}(L)|_{U_\alpha} = \mathcal{K}_{n|U_\alpha}^{MW} \otimes \lambda_\alpha \rightarrow \mathcal{K}_{n|U_\alpha}^{MW} \otimes \lambda_\alpha \otimes \mu_\alpha^{\otimes 2}$$

sending $x \otimes \lambda_\alpha$ to $x \otimes \lambda_\alpha \otimes \mu_\alpha^{\otimes 2}$. That this map gives a well-defined map

$$\psi_M : \mathcal{K}_n^{MW}(L) \rightarrow \mathcal{K}_n^{MW}(L \otimes M^{\otimes 2})$$

follows from the relation $\langle \xi_{\alpha\beta}^L \rangle = \langle \xi_{\alpha\beta}^L (\xi_{\alpha\beta}^M)^2 \rangle$ in $\mathcal{GW}(U_\alpha \cap U_\beta)$.

Let $\pi : L \rightarrow X$ be a line bundle on a smooth k -scheme X and let t be the canonical section of π^*L . If we chose a trivializing open cover $\mathcal{U} = \{U_\alpha\}$ for L on X , with local generating section λ_α on $L|_{U_\alpha}$ and transition functions $\xi_{\alpha\beta}, \lambda_\beta = \xi_{\alpha\beta} \cdot \lambda_\alpha$, then we can write t as

$$t := t_\alpha \cdot \lambda_\alpha$$

on $\pi^{-1}(U_\alpha)$, satisfying of course

$$t_\alpha = \xi_{\alpha\beta} t_\beta.$$

We have as well the class $\text{th}(L)^0 \in H^0(X, \pi_* \mathcal{K}_1^{MW}(L^{-1}))$ with $\partial \text{th}(L)^0 = \text{th}(L)$ in $H_{s_0(X)}^1(L, \mathcal{K}_1^{MW}(L^{-1}))$.

Lemma 8.2. *The restriction of $\text{th}(L)^0$ to U_α is represented by the section $[t_\alpha] \otimes \lambda_\alpha^{-1} \in \mathcal{K}_1^{MW}(L^{-1})(\pi^{-1}U_\alpha \setminus 0_Y)$*

Proof. This follows directly from the definition of $\text{th}(L)^0$. \square

Following Morel, for a non-negative integer n , we let n_ϵ denote the element $\sum_{i=0}^{n-1} \langle -1 \rangle^i$ of $\mathcal{K}_0^{MW}(Y)$ for any k -scheme Y .

Proposition 8.3. *Take $Y \in \mathbf{Sm}/k$, let $\pi : L \rightarrow Y$ be a line bundle with 0-section s_0 and let $s : Y \rightarrow L$ be a non-zero section. Let Z be the zero-locus of s and let z be a generic point of Z . Let $\lambda \in L \otimes \mathcal{O}_{Y,z}$ be a generating section of L in a neighborhood of z , let $t_z \in \mathfrak{m}_z \subset \mathcal{O}_{Y,z}$ be a uniformizing parameter and write*

$$s = u_z t_z^{n_z} \cdot \lambda_z$$

for $u_z \in \mathcal{O}_{Y,z}^\times$ and $n_z \in \mathbb{Z}_{\geq 0}$. Let $e_z(L; s)$ be the restriction of $e_Z(L; s)$ to $H_z^1(\text{Spec } \mathcal{O}_{Y,z}, \mathcal{K}_1^{MW}(L^{-1}))$ and let \bar{u}_z be the image of u_z in $k(z)$. Then under the identification

$$H_z^1(\text{Spec } \mathcal{O}_{Y,z}, \mathcal{K}_1^{MW}(L^{-1})) \cong K_0^{MW}((\mathfrak{m}_z/\mathfrak{m}_z^2)^\vee \otimes L^{-1})(k(z))$$

given by the Gersten resolution, we have

$$e_z(L; s) = \langle \bar{u}_z \rangle n_\epsilon \otimes \partial / \partial t_z \otimes \lambda_z^{-1}.$$

Proof. Take a trivializing open subset $U \subset Y$ containing z such that λ generates L over U and let s_U be the restriction of s to U . Since $\partial \text{th}(L)^0 = \text{th}(L)$ and $e_Z(L; s) = s^* \text{th}(L)$, we have

$$e_{Z \cap U}(L; s) = \partial_Z(s_U^* \text{th}(L)^0) \in H_{Z \cap U}^2(U, \mathcal{K}_1^{MW}(L^{-1})),$$

where ∂_Z is the boundary map in the local cohomology sequence

$$H^1(U \setminus Z, \mathcal{K}_1^{MW}(L^{-1})) \xrightarrow{\partial_Z} H_{Z \cap U}^2(U, \mathcal{K}_1^{MW}(L^{-1})).$$

We use the Gersten complex to compute $s_U^* \text{th}(L)^0$ and $\partial_Z(s_U^* \text{th}(L)^0)$. Let $t_U \otimes \lambda$ be the restriction of the canonical section t to $\pi^{-1}U$. Since $\text{th}(L)^0_{\pi^{-1}U} = [t_U] \otimes \lambda^{-1}$, the relevant term $\partial_Z s_U^* \text{th}(L)^0$ in $\partial_Z(s_U^* \text{th}(L)^0)$ is given by

$$\begin{aligned} \partial_Z s_U^* \text{th}(L)^0 &= \partial_Z([u_z t_z^{n_z}] \otimes \lambda^{-1}) \\ &= \partial_{t_z}([u_z t_z^{n_z}]) \otimes \partial / \partial t_z \otimes \lambda_z^{-1} \\ &= \partial_{t_z}([u_z] + \langle u_z \rangle n_\epsilon [t_z]) \otimes \partial / \partial t_z \otimes \lambda_z^{-1} \\ &= \langle \bar{u}_z \rangle n_\epsilon \otimes \partial / \partial t_z \otimes \lambda_z^{-1}. \end{aligned}$$

Here ∂_{t_z} is the boundary map $K_1^{MW}(k(Y)) \rightarrow K_0^{MW}(k(z))$ associated to the parameter $t_z \in \mathcal{O}_{Y,z}$, and we use the relation

$$[u_z t_z^{n_z}] = [u_z] + \langle u_z \rangle n_\epsilon [t_z]$$

in $K_1^{MW}(k(Y))$, the fact that ∂_{t_z} is a $\mathcal{K}_0^{MW}(\mathcal{O}_{Y,z})$ -module map and the relation $\partial_{t_z}([t_z]) = 1$. \square

One can also give a description of the global Euler class $e(L)$ in terms of cocycles. For this, choose a trivializing open cover $\mathcal{U} := \{U_\alpha\}$ for L with isomorphisms $\phi_\alpha : \mathcal{O}_{U_\alpha} \rightarrow L|_{U_\alpha}$ and let $\lambda_\alpha \in L(U_\alpha)$ be the corresponding generating section $\phi_\alpha(1)$. Let $\xi_{\alpha\beta} \in \mathcal{O}_Y^\times(U_\alpha \cap U_\beta)$ be the resulting transition functions determined by

$$\lambda_\beta = \xi_{\alpha\beta} \lambda_\alpha$$

over $U_\alpha \cap U_\beta$.

Lemma 8.4. *The image of $\text{th}(L)$ in $H^1(L, \mathcal{K}_1^{MW}(L^{-1}))$ under the “forget supports” map*

$$H_{0Y}^1(L, \mathcal{K}_1^{MW}(L^{-1})) \rightarrow H^1(L, \mathcal{K}_1^{MW}(L^{-1}))$$

is represented by the cocycle $\{[\xi_{\alpha\beta}] \otimes \lambda_\alpha^{-1} \in \mathcal{K}_1^{MW}(\pi^ L^{-1})(V_\alpha \cap V_\beta)\}_{\alpha\beta}$ for the cover $\mathcal{V} := \{V_\alpha := \pi^{-1}(U_\alpha)\}$ of L .*

Proof. Defining the sheaf $\bar{\mathcal{K}}_1^{MW}(L^{-1})$ on V by the exact sheaf sequence

$$0 \rightarrow \mathcal{K}_1^{MW}(L^{-1})_V \rightarrow j_* \mathcal{K}_1^{MW}(L^{-1})_{V \setminus 0_Y} \rightarrow \bar{\mathcal{K}}_1^{MW}(L^{-1}) \rightarrow 0,$$

we have $\bar{R}^0 p_* \mathcal{K}_1^{MW}(L^{-1}) = p_* \bar{\mathcal{K}}_1^{MW}(L^{-1})$. Comparing with the distinguished triangle

$$R s_0^! \mathcal{K}_1^{MW}(L^{-1})_V \rightarrow \mathcal{K}_1^{MW}(L^{-1})_V \rightarrow R j_* \mathcal{K}_1^{MW}(L^{-1})_{V \setminus 0_Y}$$

shows that the image of $\text{th}(L)$ in $H^1(L, \mathcal{K}_1^{MW}(L^{-1}))$ is given by applying the boundary map

$$\tilde{\partial} : H^0(V, \bar{\mathcal{K}}_1^{MW}(L^{-1})) \rightarrow H^1(V, \mathcal{K}_1^{MW}(L^{-1}))$$

to $-\text{th}(V)^0 \in H^0(V, \bar{\mathcal{K}}_1^{MW}(L^{-1}))$.

Since $\text{th}(L)^0$ is represented by

$$\{[t_\alpha] \otimes \lambda_\alpha^{-1} \text{ on } V_\alpha \setminus 0_Y\}_\alpha,$$

it follows that $\tilde{\partial}(-\text{th}(L)^0)$ is represented by the cocycle

$$\{\text{res}_{V_\alpha \cap V_\beta}([t_\alpha] \otimes \lambda_\alpha^{-1}) - \text{res}_{V_\alpha \cap V_\beta}([t_\beta] \otimes \lambda_\beta^{-1}) \text{ on } V_\alpha \cap V_\beta\}_{\alpha\beta}.$$

Since

$$\begin{aligned} [t_\alpha] \otimes \lambda_\alpha^{-1} &= [\xi_{\alpha\beta} t_\beta] \otimes \lambda_\alpha^{-1} \\ &= ([\xi_{\alpha\beta}] + \langle \xi_{\alpha\beta} \rangle [t_\beta]) \otimes \lambda_\alpha^{-1} \\ [t_\beta] \otimes \lambda_\beta^{-1} &= [t_\beta] \otimes \xi_{\alpha\beta} \lambda_\alpha^{-1} \\ &= \langle \xi_{\alpha\beta} \rangle [t_\beta] \otimes \lambda_\alpha^{-1} \end{aligned}$$

after restriction to $V_\alpha \cap V_\beta$, this cocycle is simply

$$\{[\xi_{\alpha\beta}] \otimes \lambda_\alpha^{-1} \text{ on } V_\alpha \cap V_\beta\}_{\alpha\beta}.$$

□

Proposition 8.5. *Let $\pi : L \rightarrow Y$ be a line bundle on some $Y \in \mathbf{Sm}/k$, with trivializing open cover $\mathcal{U} = \{U_\alpha\}$, local generating sections $\lambda_\alpha \in L(U_\alpha)$ and transition functions $\xi_{\alpha\beta} \in \mathcal{O}_Y^\times(U_\alpha \cap U_\beta)$, $\lambda_\beta = \xi_{\alpha\beta} \lambda_\alpha$. Then $e(L) \in H^1(Y, \mathcal{K}_1^{MW}(L^{-1}))$ is represented by the cocycle*

$$\{[\xi_{\alpha\beta}] \otimes \lambda_\alpha^{-1} \in \mathcal{K}_0^{MW}(L^{-1})(U_\alpha \cap U_\beta)\}_{\alpha\beta}.$$

for the cover \mathcal{U} .

Proof. This follows from the fact that $s_{0*}(1_Y) \in H^1(L, \mathcal{K}_1^{MW}(\pi^* L^{-1}))$ is the image of $\text{th}(L)$ under the forget supports map, the identity $e(L) = s_0^* s_{0*}(1_Y)$ and Lemma 8.4. □

Corollary 8.6. *Let L be a line bundle on a smooth k -scheme X . We have the canonical isomorphism (see Remark 8.1)*

$$\psi_L : \mathcal{K}_1^{MW}(L^{-1}) \rightarrow \mathcal{K}_1^{MW}(L).$$

Then

$$e(L^{-1}) = -\langle -1 \rangle \psi_L(e(L)).$$

Proof. Choose a trivializing open cover $\mathcal{U} = \{U_\alpha\}$ for L , with local generating sections $\lambda_\alpha \in L(U_\alpha)$ satisfying $\lambda_\beta = \xi_{\alpha\beta} \lambda_\alpha$ over $U_\alpha \cap U_\beta$. This gives the local generating section $\lambda_\alpha^{-1} \in L^{-1}(U_\alpha)$ with $\lambda_\beta^{-1} = \xi_{\alpha\beta}^{-1} \lambda_\alpha^{-1}$ over $U_\alpha \cap U_\beta$. Applying the isomorphism

$$\psi_L : H^1(X, \mathcal{K}_1^{MW}(L^{-1})) \rightarrow H^1(X, \mathcal{K}_1^{MW}(L)),$$

Proposition 8.5 says that $\psi_L(e(L))$ is represented by the cocycle

$$\{[\xi_{\alpha\beta}] \otimes \lambda_\alpha \in \mathcal{K}_0^{MW}(L)(U_\alpha \cap U_\beta)\}_{\alpha\beta}$$

while $e(L^{-1})$ is represented by the cocycle

$$\{[\xi_{\alpha\beta}^{-1}] \otimes \lambda_\alpha \in \mathcal{K}_0^{MW}(L)(U_\alpha \cap U_\beta)\}_{\alpha\beta}.$$

The result follows from this and the identity

$$[u^{-1}] = -\langle -1 \rangle [u]$$

in $\mathcal{K}_1^{MW}(\mathcal{O})$ for $u \in \mathcal{O}^\times$. □

Remark 8.7. Let $h = 1 + \langle -1 \rangle \in \text{GW}(k)$ be the class of the standard hyperbolic form. Since $h\eta = 0$ in $K_{-1}^{MW}(k)$ and $K_*^M \cong K_*^{MW}/\langle \eta \rangle$, multiplication by h on K_*^{MW} descends to the hyperbolic map

$$\bar{h} : K_*^M(F) \rightarrow K_*^{MW}(F).$$

In fact, for L a line bundle on a smooth k -scheme Y , coming from a trivializing open cover $\mathcal{U} = \{U_\alpha\}$ and cocycle $\{\xi_{\alpha\beta}\}$, the identity $h\langle \xi_{\alpha\beta} \rangle = h$

shows that the hyperbolic map extends to a hyperbolic map of sheaves on Y

$$\bar{h} : \mathcal{K}_*^M \rightarrow \mathcal{K}_*^{MW}(L),$$

factoring the map $\times h : \mathcal{K}_*^{MW}(L) \rightarrow \mathcal{K}_*^{MW}(L)$ through the quotient map $q : \mathcal{K}_*^{MW}(L) \rightarrow \mathcal{K}_*^M$.

For $Z \subset Y$ a closed subset containing no generic point of Y , the Gersten resolution for \mathcal{K}_1^M gives a canonical identification of $H_Z^1(Y, \mathcal{K}_1^M)$ with $\text{div}_Z(Y)$, the group of divisors on Y with support contained in Z . If $s : Y \rightarrow L$ is a section of a line bundle $L \rightarrow Y$, with divisor $\text{div}(s)$ supported in Z , then the image of $\text{div}(s) \in \text{div}_Z(Y) \cong H_Z^1(Y, \mathcal{K}_1^M)$ in $H^1(Y, \mathcal{K}_1^M)$ under the forget the support map is just the usual first Chern class $c_1(L) \in H^1(Y, \mathcal{K}_1^M)$. We denote $\text{div}(s)$, considered as an element of $H_Z^1(Y, \mathcal{K}_1^M)$, by $c_{1Z}(L; s)$. Since $\text{div}(s) = s^*(0_Y)$, it is not hard to see that

$$c_1(L) = q(e(L)); \quad c_{1Z}(L; s) = q(e_Z(L; s)).$$

As a particular consequence of Proposition 8.3, we see that, if $L = M^{\otimes 2}$ for some line bundle M on Y and $s = uw^{\otimes 2}$ for some section w of M and some unit u on Y , then

$$(8.1) \quad e_Z(L; s) = \bar{h} \cdot c_{1Z}(M; w) = h \cdot e_Z(M; s).$$

Indeed, if $w_\alpha = u_z \cdot t_z^n \cdot \rho_\alpha$ is the local description of w near $z \in U_\alpha$, then $s_\alpha = u \cdot u_z^2 t_z^{2n} \cdot \lambda_\alpha$ ($\lambda_\alpha := \rho_\alpha^2$) and thus at z

$$e_z(L; s) = \langle \bar{u} \rangle \cdot n_\epsilon h \otimes \lambda_\alpha^{-1} \otimes \partial / \partial t_x$$

since $(2n)_\epsilon = n_\epsilon \cdot h$. But as $\langle v \rangle h = h$ for all units v , we see that the expression at the right is just $h \cdot \text{ord}_z w_\alpha \otimes \lambda_\alpha^{-1} \otimes \partial / \partial t_x$, which verifies the first equality in (8.1). The second equality follows from the identities

$$h \cdot e_Z(M; s) = \bar{h} \cdot q(e_Z(M; s)) = \bar{h} \cdot c_{1Z}(M; w).$$

We have a similar formula for the usual Euler class $e(L)$. If $L \cong M^{\otimes 2}$ for some line bundle M on Y , then we can use a cocycle $\{\tau_{\alpha\beta}\}_{\alpha\beta}$ description for M to give the cocycle $\{\xi_{\alpha\beta} = \tau_{\alpha\beta}^2\}_{\alpha\beta}$ for L . Then $e(L)$ is given by the cocycle

$$\{[\xi_{\alpha\beta}] \otimes \lambda_\alpha^{-1} = h \cdot [\tau_{\alpha\beta}] \otimes \lambda_\alpha^{-1}\}_{\alpha\beta},$$

which gives us the identity

$$e(L) = \bar{h} \cdot c_1(M)$$

in $H^1(Y, \mathcal{K}_1^{MW}(L^{-1}))$.

Example 8.8. Let $\pi : X \rightarrow \text{Spec } k$ be a smooth projective geometrically integral curve over k of genus g . Then

$$\chi(X) = (1 - g) \cdot h$$

in $\text{GW}(k)$. Indeed, we know that $\chi(X)$ is hyperbolic by Theorem 7.1 and the rank of $\chi(X)$ is the usual Euler characteristic $2 - 2g$. Using Corollary 8.6, this shows

$$\pi_*(e(\omega_{X/k})) = -\langle -1 \rangle (1 - g) \cdot h = (g - 1) \cdot h.$$

9. LOCAL CONTRIBUTIONS

We consider the problem of computing the Euler class with support associated to a section s of vector bundle $\pi : V \rightarrow X$ on a smooth k -scheme X . Kass and Wickelgren [21] have computed the local contributions for an arbitrary section with isolated zeros (assuming V has rank equal to the dimension of X), showing that the formula of Eisenbud-Khimshiashvili-Levine, originally for sections of real bundles on an oriented manifold, computes the local Euler class in general. Here we give a much more restricted treatment, an elementary extension of the one-dimensional case, which however is still useful for many computations.

We will assume

(9.1)

- (1) V has rank $d = \dim_k X$.
- (2) s has isolated zeros
- (3) s is “locally diagonalizable”: Suppose $x \in X$ is a zero of s . Then there is a basis of sections of V near x , $\lambda_1, \dots, \lambda_d$, and a system of uniformizing parameters $t_1, \dots, t_d \in \mathfrak{m}_x$ such that, if we write

$$s = \sum_{i=1}^d s_i \lambda_i : \mathcal{O}_{X,x} \rightarrow V \otimes \mathcal{O}_{X,x},$$

then there are units $u_i \in \mathcal{O}_{X,x}^\times$ and integers $n_i \geq 1$ such that

$$s_i \equiv u_i t_i^{n_i} \pmod{\mathfrak{m} \cdot (t_1^{n_1}, \dots, t_d^{n_d})}$$

We let $\wedge \lambda_*^{-1}$ denote the generator $\lambda_1^{-1} \wedge \dots \wedge \lambda_d^{-1}$ of $\det^{-1} V \otimes \mathcal{O}_{X,x}$ and let $\wedge \partial / \partial t_*$ denote the generator $\partial / \partial t_1 \wedge \dots \wedge \partial / \partial t_d$ of $\det^{-1} \mathfrak{m}_x / \mathfrak{m}_x^2$.

Let $Z := s^{-1}(0_X)_{\text{red}} \subset X$. Under the assumptions (1) and (2), the Euler class $e_Z(V; s) \in H_Z^d(X, \mathcal{K}_d^{MW}(\det^{-1} V))$ is a sum

$$e_Z(V; s) = \sum_{x \in Z} e_x(V; s).$$

Here $e_x(V; s)$ is the component of $e_Z(V; s)$ in $H_x^d(X, \mathcal{K}_d^{MW}(\det^{-1} V))$ under the excision decomposition

$$H_Z^d(X, \mathcal{K}_d^{MW}(\det^{-1} V)) = \oplus_{x \in Z} H_x^d(X, \mathcal{K}_d^{MW}(\det^{-1} V)).$$

Via the Gersten complex for $\mathcal{K}_d^{MW}(\det^{-1}(V))$, we have the identification

$$H_x^d(X, \mathcal{K}_d^{MW}(\det^{-1} V)) = \mathcal{K}_0^{MW}(\det^{-1}(V) \otimes \det^{-1}(\mathfrak{m}_x / \mathfrak{m}_x^2))(k(x)).$$

Proposition 9.1. *Under the assumptions (9.1), let x be a closed point of X with $s(x) = 0$, and let \bar{u}_i be the image of u_i in $k(x)^\times$. Then $e_x(V; s) \in \mathcal{K}_0^{MW}(\det^{-1}(V) \otimes \det^{-1}(\mathfrak{m}_x / \mathfrak{m}_x^2))(k(x))$ is given by*

$$e_x(V; s) = (\langle \prod_{i=1}^d \bar{u}_i \rangle \cdot \prod_{i=1}^d (n_i)_\epsilon) \otimes \wedge \lambda_*^{-1} \otimes \wedge \partial / \partial t_*.$$

Proof. We first reduce to the case in which X is a $k(x)$ -scheme. For this, we consider the base-change $\pi : X_{k(x)} \rightarrow X$. The $k(x)$ -point x lifts to a $k(x)$ -point \tilde{x} of $X_{k(x)}$ and π induces an isomorphism

$$\pi^* : H_x^d(X, \mathcal{K}_d^{MW}(\det^{-1} V)) \rightarrow H_{\tilde{x}}^d(X_{k(x)}, \mathcal{K}_d^{MW}(\det^{-1} \pi^* V))$$

with $\pi^*(e_x(V; s)) = e_{\tilde{x}}(\pi^* V; \pi^* s)$. Thus, we may assume that X is a $k(x)$ -scheme and in particular $k(x) \subset \mathcal{O}_{X,x}$.

Next, we reduce to the case of a diagonal section, that is, a section of the form $s = \sum_{i=1}^d a_i t_i^{n_i} \lambda_i$, with $a_i \in k(x)$. For this, we may replace X with any convenient neighborhood of x in X ; in particular, we may assume that the λ_i form a basis of sections for V over X . Consider the given section $s = \sum_{i=1}^d s_i \lambda_i$ and let $a_i = \bar{u}_i \in k(x)$. Let s' be the section $\sum_{i=1}^d a_i t_i^{n_i} \lambda_i$ of V in some neighborhood of x . By assumption $s_i = a_i t_i^{n_i}$ modulo $\mathfrak{m} \cdot (t_1^{n_1}, \dots, t_d^{n_d})$. Let T be the standard coordinate on $\mathbb{A}_{k(x)}^1 = \text{Spec } k(x)[T]$. Consider the vector bundle $p_1^* V$ on $X \times_{k(x)} \mathbb{A}_{k(x)}^1$ and the section

$$\tilde{s} := (1 - T) \cdot s + T \cdot s'.$$

Since \tilde{s} is a section in $p_1^* V \otimes (t_1^{n_1}, \dots, t_d^{n_d}) \mathcal{O}_{X,x}[T]$ and $s \equiv s'$ modulo $\mathfrak{m}(t_1^{n_1}, \dots, t_d^{n_d}) \mathcal{O}_{X,x}[T]$, it follows that the zero-locus \tilde{Z} of \tilde{s} (restricted to $\text{Spec } \mathcal{O}_{X,x}[T]$) is $x \times_{k(x)} \mathbb{A}_{k(x)}^1 \subset \text{Spec } \mathcal{O}_{X,x}[T]$. The \mathbb{A}^1 -invariance of Milnor-Witt cohomology thus implies that

$$e_x(V; s) = i_0^* e_{x \times \mathbb{A}^1}(\pi^* V; \tilde{s}) = i_1^* e_{x \times \mathbb{A}^1}(\pi^* V; \tilde{s}) = e_x(V; s').$$

We now reduce to the case $X = \mathbb{A}_k^d$, $x = (0, \dots, 0)$ and $V = \bigoplus_{i=1}^d p_i^* \mathcal{O}_{\mathbb{A}_k^1}$, where $p_i : \mathbb{A}^d \rightarrow \mathbb{A}^1$ is the i th projection, and λ_i is the corresponding i th basis element $p_i^*(1)$. Indeed, changing notation, we may assume that $k(x) = k$. The coordinates (t_1, \dots, t_d) give an étale morphism

$$(t_1, \dots, t_d) : (X, x) \rightarrow (\mathbb{A}^d, 0^d).$$

The basis $(\lambda_1, \dots, \lambda_d)$ of $V \otimes \mathcal{O}_{X,x}$ gives the isomorphism

$$V \otimes \mathcal{O}_{X,x} \cong \mathcal{O}_{X,x}^d \cong (t_1, \dots, t_d)^*(\bigoplus_i p_i^* \mathcal{O}_{\mathbb{A}^1, 0})$$

sending λ_i to $p_i^*(1)$; call this isomorphism ψ . Similarly, if we use the standard coordinates T_1, \dots, T_d for \mathbb{A}^d , we have

$$\sum_i a_i t_i^{n_i} \lambda_i = \psi((t_1, \dots, t_d)^*(\sum_i p_i^*(a_i T_i^{n_i}))).$$

As (t_1, \dots, t_d) and ψ induces the isomorphism

$$\psi^{-1} \circ (t_1, \dots, t_d)^* : H_{0^d}^d(\mathbb{A}^d, \mathcal{K}_d^{MW}) \rightarrow H_x^d(X, \mathcal{K}_d^{MW}(\det^{-1} V))$$

with

$$\begin{aligned} \psi^{-1} \circ (t_1, \dots, t_d)^* (\langle \prod_{i=1}^d a_i \rangle \cdot \prod_{i=1}^d (n_i)_\epsilon \otimes \wedge \partial / \partial T_*) \\ = \langle \prod_{i=1}^d a_i \rangle \cdot \prod_{i=1}^d (n_i)_\epsilon \otimes \wedge \lambda_*^{-1} \otimes \wedge \partial / \partial t_*, \end{aligned}$$

we have achieved the desired reduction.

In the case of the section

$$s = (a_1 T_1^{n_1}, \dots, a_d T_d^{n_d})$$

of $O_{\mathbb{A}^d, 0}^d$, we may use Lemma 5.7. This gives

$$e_{0^d; s}(O_{\mathbb{A}^d, 0}^d) = \cup_{i=1}^d p_i^* e_{0, a_i T^{n_i}}(O_{\mathbb{A}^1, 0}).$$

Since

$$e_{0, a_i T^{n_i}}(O_{\mathbb{A}^1, 0}) = \langle a_i \rangle (n_i)_\epsilon \otimes \partial / \partial T_i$$

by Proposition 8.3, the result follows. \square

Example 9.2. We consider the simplest case of a vector bundle $\pi : V \rightarrow X$ of rank $d = \dim_k X$ with a section $s : X \rightarrow V$ which is transverse to the zero-section. If $x \in X$ is a zero of s , choose a basis of sections $\lambda_1, \dots, \lambda_d$ of V in a neighborhood of x , and a system of parameters $t_1, \dots, t_d \in \mathfrak{m}_x$. As in the proof of Proposition 9.1, we may assume that $k(x) = k$. If we write s as $s = \sum_{i=1}^d s_i \lambda_i$, the condition that s is transverse to the zero-section at x translates in the fact that the matrix

$$\partial s / \partial t := (\partial s_i / \partial t_j) \in M_{d \times d}(k(x))$$

is invertible. Replacing $\lambda_1, \dots, \lambda_d$ with a new basis $\lambda'_1, \dots, \lambda'_d$ and replacing the parameters t_1, \dots, t_d by a new set of parameters t'_1, \dots, t'_d by a $k(x)$ -linear isomorphism, we may assume that $\partial s / \partial t$ is the identity matrix, in particular, that the section s is locally diagonalizable. Proposition 9.1 gives

$$e_x(V; s) = \langle 1 \rangle \otimes \wedge \lambda_*'^{-1} \otimes \wedge \partial / \partial t'_*.$$

Keeping track of what the change of coordinates does to the generator $\wedge \lambda_*^{-1}$ for $\det^{-1} V_x$ and the generator $\wedge \partial / \partial t_*$ for $\det^{-1} \mathfrak{m}_x / (\mathfrak{m}_x)^2$, we have

$$(9.2) \quad e_x(V; s) = \langle \det(\partial s / \partial t) \rangle \otimes \wedge \lambda_*^{-1} \otimes \wedge \partial / \partial t_*.$$

Remark 9.3. Although we are restricting to the case of a bundle of rank equal to the dimension of the base-scheme, this is not an essential restriction. The local Euler class $e_{s=0}(V, s)$ for a vector bundle $V \rightarrow Y$ is determined by the restriction to $\text{Spec } \mathcal{O}_{Y, y}$ for all generic points y of $(s = 0)$. Thus, we can reduce to the case $\text{rank}(V) = \dim Y$ if the section s has zero-locus of codimension equal to the rank of V . In this case, the work of Kass-Wickelgren [21] applies to compute $e_y(V; s)$, and if we assume the local diagonalizability property near each y , we may also use the method presented here.

10. TWISTING A BUNDLE BY A LINE BUNDLE

With the help of Theorem 1, we have quite a few tools at hand to compute the Euler characteristic of T_X , for X a smooth projective variety. Unfortunately, enumerative geometry uses other bundles as input. In contrast to the theory of Chern classes, the Euler class does not in general come along with other “lower” classes, or a splitting principle, so one does not have as large a toolbox for computations as for Chern classes. One does have a replacement for the Whitney product formula, namely, for

$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

an exact sequence of vector bundles on some smooth scheme X , we have

$$e(V) = e(V')e(V'') \in H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$$

where r is the rank of V . However, the usual formulas for associated bundles, like tensor products, exterior or symmetric powers, duals, and the like, are all lacking.

We consider the following situation. Let X be a smooth, quasi-projective k -scheme with a vector bundle V of rank r and a line bundle L . Our goal is to relate the Euler classes $e(L)$, $e(V)$ and $e(V \otimes L)$.

As the Euler classes are homotopy invariant, we may replace X with L , and, changing notation, we may assume without loss of generality that L admits a global section $s : \mathcal{O}_X \rightarrow L$ with divisor D a pure codimension one closed subscheme of X . Using s , we have the map of X -schemes

$$\phi_s : V \rightarrow L \otimes V$$

sending v to $s(1) \otimes v$; over $X \setminus D$, ϕ_s is an isomorphism. Let $\pi_V : V \rightarrow X$, $\pi_{L \otimes V} : L \otimes V \rightarrow X$ be the projections and $s_0^V : X \rightarrow V$, $s_0^{L \otimes V} : X \rightarrow L \otimes V$ the zero-sections.

We write L^n for the n th tensor power of L . As in §4.1, we have the Thom class

$$\text{th}(L \otimes V) \in H_{s_0^{L \otimes V}(X)}^r(L \otimes V, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)),$$

which we may pullback by ϕ_s to give the class

$$\phi_s^*(\text{th}(L \otimes V)) \in H_{s_0^V(X) \cup \pi_V^{-1}(D)}^r(V, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)).$$

Here we are writing $\mathcal{K}_r^{MW}(\det^{-1} V)$ for the sheaf $\mathcal{K}_r^{MW}(\pi_V^*(\det^{-1} V))$ on V and similarly for the sheaf $\mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)$ on $L \otimes V$.

Let $F = s_0^V(X) \cap \pi_V^{-1}(D)$.

Lemma 10.1. *The restriction map*

$$\begin{aligned} H_{s_0^V(X) \cup \pi_V^{-1}(D)}^r(V, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \\ \rightarrow H_{s_0^V(X) \cup \pi_V^{-1}(D) \setminus F}^r(V \setminus F, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \end{aligned}$$

is injective.

Proof. We may compute both cohomology groups using the Gersten complexes on V and on $V \setminus F$, with support in $s_0^V(X) \cup \pi_V^{-1}(D)$ and $s_0^V(X) \cup \pi_V^{-1}(D) \setminus F$, respectively. Since F has codimension $r+1$ on V , the restriction map is an isomorphism on the terms in degrees r and $r-1$, whence the result. \square

Let $j : V \setminus F \rightarrow V$ be the inclusion. Since $s_0^V(X) \cup \pi_V^{-1}(D) \setminus F = (s_0^V(X) \setminus F) \amalg (\pi_V^{-1}(D) \setminus F)$, we may write $j^* \phi_s^*(\text{th}(L \otimes V))$ as the sum of two classes

$$j^* \phi_s^*(\text{th}(L \otimes V)) = \phi_s^*(\text{th}(L \otimes V))_{s_0} + \phi_s^*(\text{th}(L \otimes V))_D$$

with

$$\begin{aligned} \phi_s^*(\text{th}(L \otimes V))_{s_0} &\in H_{s_0^V(X) \setminus F}^r(V \setminus F, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)), \\ \phi_s^*(\text{th}(L \otimes V))_D &\in H_{\pi_V^{-1}(D) \setminus F}^r(V \setminus F, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)). \end{aligned}$$

We first discuss the class $\phi_s^*(\text{th}(L \otimes V))_{s_0}$. By excision the restriction map

$$\begin{aligned} H_{s_0^V(X) \setminus F}^r(V \setminus F, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \\ \rightarrow H_{s_0^V(X) \setminus F}^r(V \setminus \pi_V^{-1}(D), \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \end{aligned}$$

is an isomorphism. Let $U = X \setminus D$ with inclusion $j_U : U \rightarrow X$ and let $V_U \rightarrow U$ be the restriction of V to U . We may consider $\phi_s^*(\text{th}(L \otimes V))_{s_0}$ as an element of $H_{s_0^V(U)}^r(V_U, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$. Over U , the section s gives an isomorphism of O_U with L and the map $\phi_s : V_U \rightarrow (L \otimes V)_U$ is an isomorphism. We have the isomorphism

$$s_U^{\otimes -r*} : H_{s_0^V(U)}^r(V_U, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \rightarrow H_{s_0^V(U)}^r(V_U, \mathcal{K}_r^{MW}(\det^{-1} V_U))$$

and the functoriality of Thom classes gives the identity

$$(10.1) \quad s_U^{\otimes -r*}(\phi_s^*(\text{th}(L \otimes V))_{s_0}) = \text{th}(V_U)$$

in $H_{s_0^V(U)}^r(V_U, \mathcal{K}_r^{MW}(\det^{-1} V_U))$.

Now, for the class $\phi_s^*(\text{th}(L \otimes V))_D$. We again localize further. Letting $j_{s_0} : V \setminus s_0^V(X) \rightarrow V$ be the inclusion, the restriction map

$$\begin{aligned} H_{\pi_V^{-1}(D) \setminus F}^r(V \setminus F, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \\ \xrightarrow{j_{s_0}^*} H_{\pi_V^{-1}(D) \setminus F}^r(V \setminus s_0(X), \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \end{aligned}$$

is an isomorphism. Over $V \setminus s_0(X)$, the canonical section of V gives us an extension of vector bundles on $V \setminus s_0(X)$

$$0 \rightarrow O_{V \setminus s_0(X)} \rightarrow j_{s_0}^* V \rightarrow W \rightarrow 0;$$

tensoring with $j_{s_0}^* \pi_V^* L$ yields the short exact sequence of vector bundles on $V \setminus s_0(X)$

$$(10.2) \quad 0 \rightarrow j_{s_0}^* \pi_V^* L \rightarrow j_{s_0}^* (\pi_V^* L \otimes V) \rightarrow \pi_V^* L \otimes W \rightarrow 0.$$

We have the Euler class

$$e(\pi_V^* L \otimes W) \in H^{r-1}(V \setminus s_0(X), \mathcal{K}_{r-1}^{MW}(L^{1-r} \otimes \det^{-1} W)).$$

Pulling back the Thom class $\text{th}(L)$ by the section s and then pulling back to V gives us the Euler class with support

$$e_{\pi^{-1}(D)}(\pi_V^* L; s_V) \in H_{\pi_V^{-1}(D)}^1(V, \mathcal{K}_1^{MW}(L^{-1})).$$

Lemma 10.2. *We have the identity*

$$\phi_s^*(\text{th}(L \otimes V))_D = e_{\pi^{-1}(D)}(\pi_V^* L; s_V) \cup e(\pi_V^* L \otimes W)$$

$$\text{in } H_{\pi_V^{-1}(D) \setminus s_0(X)}^r(V \setminus s_0(X), \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)).$$

Proof. We first note that the exact sequence (10.2) gives a canonical isomorphism $L^{-r} \otimes \det^{-1} V \cong L^{-1} \otimes L^{1-r} \otimes \det^{-1} W$ on $V \setminus s_0(X)$, which puts the cup product $e_{\pi^{-1}(D)}(\pi_V^* L; s_V) \cup e(\pi_V^* L \otimes W)$ in the cohomology group $H_{\pi_V^{-1}(D) \setminus s_0(X)}^r(V \setminus s_0(X), \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$, as asserted.

To check the identity, we may replace $V \setminus s_0(X)$ with a Jouanolou cover $q : Y \rightarrow V \setminus s_0(X)$. Splitting the pullback of the exact sequence (10.2) gives us the isomorphism $q^* \phi_s^* \pi_{L \otimes V}^* L \otimes V \cong q^* \pi_V^* L \oplus q^* \pi_V^* L \otimes W$. This in turn induces an identity of Thom classes

$$\text{th}(q^* \phi_s^* \pi_{L \otimes V}^* L \otimes V) \cong p_1^* \text{th}(q^* \pi_V^* L) \cup p_2^* \text{th}(q^* (\pi_V^* L \otimes W))$$

where

$$\begin{aligned} p_1 : q^* \pi_V^* L \oplus q^* (\pi_V^* L \otimes W) &\rightarrow q^* \pi_V^* L, \\ p_2 : q^* \pi_V^* L \oplus q^* W &\rightarrow q^* (\pi_V^* L \otimes W) \end{aligned}$$

are the projections. Pulling back by the section $(q^* \pi_V^* s, s_0)$ of $q^* \pi_V^* L \oplus q^* (\pi_V^* L \otimes W)$ gives the identity

$$\begin{aligned} q^* \phi_s^*(\text{th}(L \otimes V))_D &= (q^* \pi_V^* s, s_0)^*(\text{th}(q^* \phi_s^* \pi_{L \otimes V}^* L \otimes V)) \\ &= q^* \pi_V^*(s^*(\text{th}(L))) \cup q^* s_0^*(\text{th}(\pi_V^* L \otimes W)) \\ &= q^*(e_{\pi^{-1}(D)}(\pi_V^* L; s_V) \cup e(\pi_V^* L \otimes W)). \end{aligned}$$

□

Remark 10.3. Lemma 10.2 makes sense and when suitably interpreted is correct even for V of rank 1. In this case, W is the 0-bundle on $V \setminus s_0(X)$, which gives the canonical isomorphism $\det W \cong \mathcal{O}_{V \setminus s_0(X)}$. In the proof of Lemma 10.2, we use the canonical isomorphism

$$L^{-r} \otimes \det^{-1} V \cong L^{-1} \otimes L^{1-r} \otimes \det^{-1} W$$

on $V \setminus s_0(X)$; in the case of rank 1, this comes down to the canonical isomorphism

$$L^{-1} \otimes \det^{-1} V \cong L^{-1}$$

given by the isomorphism $v : \mathcal{O}_{V \setminus s_0(X)} \rightarrow \pi^* V|_{V \setminus s_0(X)}$. Thus $e(\pi^* L \otimes W)$ is the element of $H^0(V \setminus s_0(X), \mathcal{K}_0^{MW}(\det^{-1} V))$ given by the symmetric isomorphism $\mathcal{O}_{V \setminus s_0(X)} \rightarrow \det^{-1} V|_{V \setminus s_0(X)}$ (with respect to the duality $\text{Hom}(-, \det^{-1} V)$).

Putting this all together gives a decomposition of $\phi_s^*(\text{th}(L \otimes V))$.

Proposition 10.4. *Let $j_F : V \setminus F \rightarrow V$ be the inclusion. Then*

$$j_F^* \phi_s^*(\text{th}(L \otimes V)) = s_U^{\otimes r^*}(\text{th}(V_U)) + e_{\pi^{-1}(D)}(\pi_V^* L; s_V) \cup e(\pi_V^* L \otimes W)$$

in $H_{(s_0(X) \cup \pi^{-1}(D)) \setminus F}^r(V \setminus F, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$.

We now suppose that one of the following two hypotheses holds:
(10.3)

- (i) V has even rank $r = 2m$
- (ii) There is a line bundle M on X with $L \cong M^{\otimes 2}$ and a section t of M with $s = t^{\otimes 2}$.

Lemma 10.5. *Let N, N' be line bundles on a smooth k -scheme Y and suppose we have an isomorphism $s : \mathcal{O}_Y \rightarrow N$. Then for each pair of integers m, r , we have $(s^{\otimes 2m})^* = \psi_{N^{\otimes -m}}$ as maps*

$$\mathcal{K}_r^{MW}(N' \otimes N^{\otimes 2m}) \rightarrow \mathcal{K}_r^{MW}(N')$$

Proof. This follows from the description of $\psi_{N^{\otimes -m}}$ in terms of local trivializations, as detailed in Remark 8.1. \square

Here is our main result for this section.

Theorem 10.6. *Suppose that (V, L) satisfies (10.3), let r be the rank of V . Suppose in addition we have a section v of V with zero-locus A and a section s of L with pure codimension one zero-locus D ; in case (ii), we assume that via the isomorphism $L \cong M^{\otimes 2}$, there is a section s_M of M with $s = s_M^{\otimes 2}$. Let $B = A \cap D$ and let*

$$\rho_D : H_{D \setminus B}^*(X \setminus B, \mathcal{K}_*^{MW}(L^{-r} \otimes \det^{-1} V)) \rightarrow H^*(X \setminus B, \mathcal{K}_*^{MW}(L^{-r} \otimes \det^{-1} V))$$

be the canonical “forget supports” map, which we may consider as a map

$$H_{D \setminus B}^*(X \setminus (A \cup B), \mathcal{K}_*^{MW}(L^{-r} \otimes \det^{-1} V)) \xrightarrow{\rho_D} H^*(X \setminus B, \mathcal{K}_*^{MW}(L^{-r} \otimes \det^{-1} V))$$

via excision. Let $j : X \setminus B \rightarrow X$ be the inclusion and let $v_0 : X \setminus A \rightarrow V \setminus 0_X$ be the restriction of v . In case (i), with V of rank $r = 2m$, we have

$$j^* e(L \otimes V) = j^* \psi_{L^{\otimes -m}}(e(V)) + \rho_D(j^* e_D(L; s) \cup v_0^* e(\pi_V^* L \otimes W))$$

in $H^r(X \setminus B, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$. In case (ii), with $L \cong M^{\otimes 2}$, we have

$$j^* e(L \otimes V) = j^* \psi_{M^{\otimes -r}}(e(V)) + \rho_D(j^* e_D(L; s_M^{\otimes 2}) \cup v_0^* e(\pi_V^* L \otimes W))$$

in $H^r(X \setminus B, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$.

Proof. We pullback the identity in Proposition 10.4 by $(s \otimes v)|_{X \setminus B}$ to yield

$$e_{(A \cup D) \setminus B}(j^* L \otimes V; j^* s \otimes v) = s_U^{\otimes r*}(e_{A \setminus B}(j^* V; j^* v)) \\ + e_{D \setminus B}(j^* L; j^* s) \cup v_0^* e(\pi_V^* L \otimes W)$$

in $H_{(A \cup D) \setminus B}^r(X \setminus B, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$. Using Lemma 10.5, we have

$$s_U^{\otimes r*}(e_{A \setminus B}(j^* V; j^* v)) = \begin{cases} \psi_{L^{\otimes -m}}(e_{A \setminus B}(j^* V; j^* v)) & \text{in case (i)} \\ \psi_{M^{\otimes -r}}(e_{A \setminus B}(j^* V; j^* v)) & \text{in case (ii)} \end{cases}$$

both identities taking place in $H_{A \setminus B}^r(X \setminus B, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$. This gives the identity in $H_{(A \cup D) \setminus B}^r(X \setminus B, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$

$$j^* e_{A \cup D}(L \otimes V; s \otimes v) \\ = \begin{cases} \psi_{L^{\otimes -m}}(j^* e_A(V; v)) \\ \quad + j^* e_D(L; s) \cup v_0^* e(\pi_V^* L \otimes W) & \text{in case (i)} \\ \psi_{M^{\otimes -r}}(j^* e_A(V; v)) \\ \quad + j^* e_D(L; s) \cup v_0^* e(\pi_V^* L \otimes W) & \text{in case (ii)}. \end{cases}$$

We can then forget the supports to yield the result. \square

Remarks 10.7. (1) Suppose V has rank $r \geq 2$. Then the line bundle $\det W$ on $V \setminus s_0(X)$ extends uniquely to a line bundle on V which is canonically isomorphic to $\pi_V^* \det V$. If V has rank 1, then $W = 0$, so $\det W$ is the trivial line bundle on $V \setminus s_0(X)$. Using the tautological section of V , we have the canonical isomorphism $\det W \cong \pi_V^* \det V$ on $V \setminus s_0(X)$, so again in this case this canonical isomorphism allows us to extend $\det W$ to a line bundle on V with a canonical isomorphism with $\pi_V^* \det V$. In the rank 1 case, we use Remark 10.3 to interpret the term $e(\pi_V^* L \otimes W)$.

(2) Suppose V has rank 2. Then W has rank one, and the canonical isomorphism $\det W \cong \pi_V^* \det V$ is just a canonical isomorphism $W \cong \pi_V^* \det V$. Thus, W extends canonically to the line bundle $\pi_V^* \det V$ on V and we achieve the identity

$$e(L \otimes V) = \psi_{L^{\otimes -1}}(e(V)) + e(L) \cup e(L \otimes \det V)$$

in $H^2(X, \mathcal{K}_2^{MW}(L^{-2} \otimes \det^{-1} V))$.

(3) Suppose that the section v in Theorem 10.6 is in “general position”, that is, the zero locus A has codimension r on X and no component of A is contained in the divisor D . Then the subset $B = A \cap D$ has codimension $r + 1$ on X and so the restriction map

$$H^r(X, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V)) \rightarrow H^r(X \setminus B, \mathcal{K}_r^{MW}(L^{-r} \otimes \det^{-1} V))$$

is injective. Thus the difference of Euler classes $e(L \otimes V) - \psi_{L^{\otimes -m}}(e(V))$ in case (i) or $e(L \otimes V) - \psi_{M^{\otimes -r}}(e(V))$ in case (ii) is uniquely determined by the identity of Theorem 10.6.

We can simplify the formula in Theorem 10.6 in the case (ii), $L \cong M^{\otimes 2}$, using the Remark 8.7. Recall that the map “multiplication by the hyperbolic form h ”

$$h \cdot : \mathcal{K}_r^{MW}(L) \rightarrow \mathcal{K}_r^{MW}(L)$$

factors through \mathcal{K}_r^M via

$$\begin{array}{ccc} \mathcal{K}_r^{MW}(L) & \xrightarrow{h \cdot (-)} & \mathcal{K}_r^{MW}(L) \\ q \downarrow & \nearrow \bar{h} \cdot (-) & \\ \mathcal{K}_r^M & & \end{array}$$

where $q : \mathcal{K}_r^{MW}(L) \rightarrow \mathcal{K}_r^M$ is the map induced by the isomorphism

$$K_*^{MW}(L)(F)/\eta \cong K_*^M(F)$$

for a field F .

Lemma 10.8. *Let V be a rank r vector bundle on $X \in \mathbf{Sm}/k$. Then*

$$h \cdot e(V) = \bar{h} \cdot c_r(V)$$

in $H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V))$, where $c_r(V) \in H^r(X, \mathcal{K}_r^M)$ is the r th Chern class of V .

Proof. We have the commutative triangle

$$\begin{array}{ccc} H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)) & \xrightarrow{h \cdot (-)} & \mathcal{K}_r^{MW} H^r(X, \mathcal{K}_r^{MW}(\det^{-1} V)) \\ q \downarrow & \nearrow \bar{h} \cdot (-) & \\ H^r(X, \mathcal{K}_r^M) & & \end{array}$$

Since $q(e(V)) = c_r(V)$, the result follows. \square

Corollary 10.9. *Let V be a rank r vector bundle on $X \in \mathbf{Sm}/k$, let M be a line bundle on X and let $L = M^{\otimes 2}$. Then*

$$e(V \otimes L) = \psi_{M^{\otimes -r}}(e(V)) + \bar{h} \cdot c_1(M) \cdot \left(\sum_{i=1}^r c_{r-i}(V) \cdot c_1(L)^{i-1} \right)$$

in $H^r(X, \mathcal{K}_r^{MW}(\det V^{-1} \otimes L^{-r}))$.

Proof. To check the identity, we may pass to a Jouanolou cover of X , so we may assume that V , L , M and $V \otimes L$ are globally generated. We may therefore assume from the beginning that we have sections v of V with zero locus A of codimension r on X , and a section s_M of M with pure codimension

one zero-locus D , such that, letting $B = A \cap D$, B has codimension $r + 1$ on X . As the restriction map

$$H^r(X, \mathcal{K}_r^{MW}(\det V^{-1} \otimes L^{-r})) \rightarrow H^r(X \setminus B, \mathcal{K}_r^{MW}(\det V^{-1} \otimes L^{-r}))$$

is injective, it suffices to check the identity on $X \setminus B$. Thus we may assume that we have sections v and s as above, with A of codimension r , D of codimension one, and with $A \cap D = \emptyset$. Let $V^0 = V \setminus 0_X$ with projection $\pi_V^0 : V^0 \rightarrow X$.

We may then apply Theorem 10.6 and reduce to showing that

$$e_D(L; s_M^{\otimes 2}) \cup v_0^* e(\pi_V^* L \otimes W) = \bar{h} \cdot c_{1D}(M) \cup \left(\sum_{i=1}^r c_{r-i}(V) \cdot c_1(L)^{i-1} \right).$$

By Remark 8.7, we have

$$e_D(L; s_M^{\otimes 2}) = h \cdot e_D(M; s_M) = \bar{h} \cdot c_{1D}(M).$$

Thus it suffices to show that

$$h \cdot e(\pi_{V^0}^* L \otimes W) = \bar{h} \cdot \pi_{V^0}^* \left(\sum_{i=1}^r c_{r-i}(V) \cdot c_1(L)^{i-1} \right)$$

in $H^{r-1}(V^0, \mathcal{K}_{r-1}^{MW}(\pi_{V^0}^*(\det^{-1} V \otimes \det^{-r+1} L)))$. By Lemma 10.8, we need only show that

$$c_{r-1}(\pi_{V^0}^* L \otimes W) = \pi_{V^0}^* \left(\sum_{i=1}^r c_{r-i}(V) \cdot c_1(L)^{i-1} \right)$$

in $H^{r-1}(V^0, \mathcal{K}_{r-1}^M)$. But on V^0 we have the exact sequence

$$0 \rightarrow \mathcal{O}_{V^0} \rightarrow \pi_{V^0}^* V \rightarrow W \rightarrow 0$$

so

$$c_i(W) = \begin{cases} \pi_{V^0}^* c_i(V) & \text{for } i = 0, \dots, r-1 \\ 0 & \text{for } i > r-1. \end{cases}$$

The standard formula for the Chern classes of a tensor product bundle thus gives

$$\begin{aligned} c_{r-1}(\pi_{V^0}^* L \otimes W) &= \sum_{i=0}^{r-1} c_{r-1-i}(W) \cdot c_1(\pi_{V^0}^* L)^i \\ &= \pi_{V^0}^* \left(\sum_{i=1}^r c_{r-i}(V) \cdot c_1(L)^{i-1} \right) \end{aligned}$$

□

11. OBSTRUCTION CLASSES AND DUAL BUNDLES

Asok and Fasel [4, Theorem 5.3.2] have given an interpretation of the Euler class in terms of obstruction theory, which we recall here. As an application, we prove the following useful result.

Theorem 11.1. *Let X be a smooth quasi-projective scheme over k , E a rank n vector bundle on X and E^\vee the dual bundle. Let*

$$\psi : H^n(X, \mathcal{K}_n^{MW}(\det^{-1} E^\vee)) \rightarrow H^n(X, \mathcal{K}_n^{MW}(\det^{-1} E))$$

be the isomorphism $\psi_{\det^{-1} E}$. Then

$$\psi(e(E^\vee)) = (-\langle -1 \rangle)^n e(E)$$

in $H^n(X, \mathcal{K}_n^{MW}(\det^{-1} E))$.

The rank 1 case is just Corollary 8.6, so in what follows, we will assume $n \geq 2$.

Recall that for a map of connected spaces $p : X \rightarrow B$ with homotopy fiber F having abelian π_1 , one has the relative Postnikov tower

$$\begin{array}{ccccccc} X & \longrightarrow & \cdots & \longrightarrow & X^{(n+1)} & \longrightarrow & X^{(n)} & \longrightarrow & X^{(n-1)} & \longrightarrow & \cdots & \longrightarrow & X^{(0)} \\ & & & & \searrow & & \downarrow & & \swarrow & & & & \nearrow \\ & & & & & & B & & & & & & \end{array}$$

p

and the homotopy cartesian squares

$$\begin{array}{ccc} X^{(n+1)} & \longrightarrow & B\pi_1(B) \\ \downarrow & & \downarrow s \\ X^{(n)} & \xrightarrow{k_{n+1}} & \hat{K}(\pi_n, n+1). \end{array}$$

Here $\hat{K}(\pi_n, n+1)$ is the total space of the bundle over $B\pi_1(B)$ with fiber $K(\pi_n(F), n)$

$$\begin{array}{ccc} K(\pi_n(F), n) & \longrightarrow & \hat{K}(\pi_n, n+1) \\ \downarrow & & \downarrow \uparrow s \\ pt & \longrightarrow & B\pi_1(B), \end{array}$$

given by the action of $\pi_1(B)$ on $K(\pi_n(F), n+1)$ through its action on $\pi_n(F)$, with section s given by the base-point in $K(\pi_n(F), n)$. The map $X^{(n+1)} \rightarrow B\pi_1(B)$ is given by the map $\pi_1 X^{(n+1)} \rightarrow \pi_1 B$.

This extends to more general model categories, such as the category of simplicial presheaves on a site with enough points, localized with respect to the site topology; we quote the result [3, Theorem 6.1.1] for the situation in the unstable \mathbb{A}^1 homotopy category. We note that this result as stated assumes that the homotopy fiber is at least 1-connected, but the method of proof also applies to the case of abelian π_1 , see for example the construction

of the Moore-Postnikov tower described in [31, §4]. We recall their result (extended to the case of abelian π_1) for the reader's convenience

Theorem 11.2 (Asok-Fasel [3, Theorem 6.1.1]). *Suppose $f : (\mathcal{E}, e) \rightarrow (\mathcal{B}, b)$ is a pointed map of \mathbb{A}^1 -connected spaces and let \mathcal{F} be the \mathbb{A}^1 -homotopy fiber of f . Assume, in addition, that f is an \mathbb{A}^1 -fibration, \mathcal{B} is \mathbb{A}^1 -local, that \mathcal{F} is \mathbb{A}^1 -connected and $\pi_1^{\mathbb{A}^1}(\mathcal{F})$ is abelian. There are pointed spaces $(\mathcal{E}(i), e_i)$, $i = 0, 1, \dots$, with $\mathcal{E}(0) = \mathcal{B}$, and commutative diagrams of pointed morphisms of pointed spaces*

$$\begin{array}{ccccc} & & \mathcal{E}^{(i+1)} & & \\ & g^{(i+1)} \nearrow & \downarrow p^{(i)} & \nwarrow h^{(i+1)} & \\ \mathcal{E} & \xrightarrow{g^{(i)}} & \mathcal{E}^{(i)} & \xrightarrow{h^{(i)}} & \mathcal{B} \end{array}$$

having the following properties:

- (i) The composite $h(i)g(i) = f$ for all $i \geq 0$.
- (ii) The morphism $\pi_n^{\mathbb{A}^1} \mathcal{E} \rightarrow \pi_n^{\mathbb{A}^1} \mathcal{E}^{(i)}$ induced by $g^{(i)}$ is an isomorphism $n \leq i$ and an epimorphism for $n = i + 1$.
- (iii) The morphism $\pi_n^{\mathbb{A}^1} \mathcal{E}^{(i)} \rightarrow \pi_n^{\mathbb{A}^1} \mathcal{B}$ induced by $h^{(i)}$ is an isomorphism for $n > i + 1$, and a monomorphism for $n = i + 1$.
- (iv) The induced map $E \rightarrow \operatorname{holim}_i \mathcal{E}^{(i)}$ is an \mathbb{A}^1 -weak equivalence.

The morphisms $p^{(i)}$ are \mathbb{A}^1 -fibrations with \mathbb{A}^1 -homotopy fiber $K(\pi_i^{\mathbb{A}^1} \mathcal{F}, i)$ for all $i \geq 0$, and $p^{(i)}$ is a twisted \mathbb{A}^1 -principal fibration. This means the following: Let

$$K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1) := E\pi^{\mathbb{A}^1} \mathcal{B} \times_{\pi_1^{\mathbb{A}^1} \mathcal{B}} K(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1),$$

where $\pi_1^{\mathbb{A}^1} \mathcal{B}$ acts on $K(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1)$ through its action on $\pi_i^{\mathbb{A}^1} \mathcal{F}$ given by the fibration $\mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{B}$. Let

$$s : B\pi_1^{\mathbb{A}^1} \mathcal{B} \rightarrow K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1)$$

be the map given by inclusion of the base point $*$ of $K(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1)$:

$$B\pi_1^{\mathbb{A}^1} \mathcal{B} = E\pi^{\mathbb{A}^1} \mathcal{B} \times_{\pi_1^{\mathbb{A}^1} \mathcal{B}} * \hookrightarrow E\pi^{\mathbb{A}^1} \mathcal{B} \times_{\pi_1^{\mathbb{A}^1} \mathcal{B}} K(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1).$$

Then there is a unique (up to \mathbb{A}^1 -homotopy) morphism

$$k_{i+1} : \mathcal{E}^{(i)} \rightarrow K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1),$$

called the k -invariant, sitting in an \mathbb{A}^1 -homotopy pullback square of the form

$$\begin{array}{ccc} \mathcal{E}^{(i+1)} & \longrightarrow & B\pi_1^{\mathbb{A}^1} \mathcal{B} \\ \downarrow & & \downarrow s \\ \mathcal{E}^{(i)} & \xrightarrow{k_{i+1}} & K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i + 1). \end{array}$$

The pair $(K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i+1), B\pi_1^{\mathbb{A}^1} \mathcal{B})$ has a cohomological interpretation, described in the classical case by [31, Proposition 3.6] and in the setting of \mathbb{A}^1 homotopy theory by Morel [26, Theorem B.3.8]. We refer the reader to [26, Appendix B, Cohomological interpretation of the obstruction sets] for verification of the various statements made below.

The action of $\pi_1^{\mathbb{A}^1} \mathcal{B}$ on $\pi_i^{\mathbb{A}^1} \mathcal{F}$ gives a sheaf of abelian groups $\tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F}$ on $B\pi_1^{\mathbb{A}^1} \mathcal{B}$, defined as

$$\tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F} := E\pi_1^{\mathbb{A}^1} \mathcal{B} \times_{\pi_1^{\mathbb{A}^1} \mathcal{B}} \pi_i^{\mathbb{A}^1} \mathcal{F}.$$

This is *a priori* a sheaf of sets, but Morel shows that it is a sheaf of abelian groups.

Given $X \in \mathbf{Sm}/k$ and morphism $\phi : X \rightarrow \mathcal{B}$ in $\mathcal{H}_{\text{Nis}}(k)$, we have the Nisnevich sheaf $\tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F}^\phi$ on X , formed by pulling back $\tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F}$ by the composition

$$X \xrightarrow{\phi} \mathcal{B} \xrightarrow{\text{can}} B\pi_1^{\mathbb{A}^1} \mathcal{B}.$$

In more concrete terms, one has the isomorphism

$$H_{\text{Nis}}^1(X, \pi_1^{\mathbb{A}^1} \mathcal{B}|_{X_{\text{Nis}}}) \cong [X, B\pi_1^{\mathbb{A}^1} \mathcal{B}]_{\mathcal{H}_{\text{Nis}}(k)}$$

so the morphism ϕ gives rise to a torsor $Y \rightarrow X$ for the sheaf of groups $\pi_1^{\mathbb{A}^1} \mathcal{B}|_{X_{\text{Nis}}}$. The sheaf $\tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F}^\phi$ is just $Y \times_{\pi_1^{\mathbb{A}^1} \mathcal{B}|_{X_{\text{Nis}}}} \pi_i^{\mathbb{A}^1} \mathcal{F}|_{X_{\text{Nis}}}$. In any case, this gives, for a pair (X, A) the cohomology $H_{\text{Nis}}^*(X, A, \tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F}^\phi)$.

For two pairs in $\mathbf{Spc}(k)$, $(\mathcal{X}, \mathcal{A})$ and $(\mathcal{Y}, \mathcal{B})$ we define

$$[(\mathcal{X}, \mathcal{A}), (\mathcal{Y}, \mathcal{B})]_{\mathcal{H}_{\text{Nis}}(k)} := \pi_0(\text{Map}(Q\mathcal{X}, R\mathcal{Y}) \times_{\text{Map}(Q\mathcal{A}, R\mathcal{Y})} \text{Map}(Q\mathcal{A}, R\mathcal{B}))$$

where $Q\mathcal{A} \rightarrow Q\mathcal{X}$ is a cofibration of cofibrant models for $\mathcal{A} \rightarrow \mathcal{X}$ and $R\mathcal{B} \rightarrow R\mathcal{Y}$ is a fibration of fibrant models for $\mathcal{B} \rightarrow \mathcal{Y}$, in the Nisnevich local model structure; we may in use just one of these two replacements to compute the π_0 : we have weak equivalences

$$\begin{aligned} & \text{Map}(\mathcal{X}, R\mathcal{Y}) \times_{\text{Map}(\mathcal{A}, R\mathcal{Y})} \text{Map}(\mathcal{A}, R\mathcal{B}) \\ & \rightarrow \text{Map}(Q\mathcal{X}, R\mathcal{Y}) \times_{\text{Map}(Q\mathcal{A}, R\mathcal{Y})} \text{Map}(Q\mathcal{A}, R\mathcal{B}) \\ & \leftarrow \text{Map}(Q\mathcal{X}, \mathcal{Y}) \times_{\text{Map}(Q\mathcal{A}, \mathcal{Y})} \text{Map}(Q\mathcal{A}, \mathcal{B}). \end{aligned}$$

We note that an element $\gamma = [(\alpha, \beta)]$ of $[(\mathcal{X}, \mathcal{A}), (\mathcal{Y}, \mathcal{B})]_{\mathcal{H}_{\text{Nis}}(k)}$ gives by restriction a map $[\alpha] : \mathcal{X} \rightarrow \mathcal{Y}$ in $\mathcal{H}_{\text{Nis}}(k)$, which we denote by $\text{res}_{\mathcal{X}} \gamma$. For $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ in $\mathcal{H}_{\text{Nis}}(k)$, we let $[(\mathcal{X}, \mathcal{A}), (\mathcal{Y}, \mathcal{B})]_{\mathcal{H}_{\text{Nis}}(k)}^\phi$ be the subset of $[(\mathcal{X}, \mathcal{A}), (\mathcal{Y}, \mathcal{B})]_{\mathcal{H}_{\text{Nis}}(k)}$ of elements γ with $\text{res}_{\mathcal{X}} \gamma = \phi$. We now give a cohomological interpretation of this group; we first recall the injective model structure on the category of cohomological complexes supported in non-positive degrees, $Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\text{Nis}}))$.

The category of unbounded complexes $Ch(Sh^{\mathbf{Ab}}(X_{\text{Nis}}))$ has the injective model structure as defined by Hovey [16, Theorem 2.2]: weak equivalences are quasi-isomorphisms, cofibrations are the injections, fibrations are surjections which are termwise split with kernel K^* a complex of injective sheaves

which is K -injective [34, 1.1, Definition]. This latter means that for each acyclic complex A^* , the complex of abelian groups $\mathrm{Hom}(A^*, K^*)^*$ is acyclic.

We have the canonical truncation functor

$$\tau^{\leq 0} : Ch(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}})) \rightarrow Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}})),$$

right adjoint to the inclusion functor

$$i : Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}})) \rightarrow Ch(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}})).$$

The injective model structure on $Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$ has weak equivalences the quasi-isomorphisms and cofibrations the injections. Fibrations are maps of the form $\tau^{\leq 0}(f)$ with f a fibration in $Ch(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$.

Lemma 11.3. *With the weak equivalences, cofibrations and fibrations as given above, $Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$ a model category. Moreover, a finite complex \bar{K}^* in $Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$ is fibrant if and only if \bar{K}^n is injective for all $n < 0$.*

Proof. This is certainly well known, but we were not able to find this statement in the literature, so we give the proof here.

The fact that $\tau^{\leq 0} \circ i = \mathrm{Id}$ enables one to verify all the axioms for a model category except that a cofibration should have the left lifting property with respect to a fibration that is a weak equivalence. This will follow easily if we can show that for every fibration and weak equivalence \bar{f} in $Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$, there is a fibration and weak equivalence \tilde{f} in $Ch(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$ with $\bar{f} = \tau^{\leq 0}\tilde{f}$.

To see this, take a fibration and weak equivalence $\bar{f} : \bar{X}^* \rightarrow \bar{Y}^*$ and write $\bar{f} = \tau^{\leq 0}f$ for $f : X^* \rightarrow Y^*$ a fibration in $Ch(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$. Since f is a termwise split surjection, $\bar{f} = \tau^{\leq 0}f$ is a surjection, and thus $\bar{K}^* := \ker \bar{f} = \tau^{\leq 0}K^*$ is acyclic. Since f is a fibration, $\bar{K}^n = K^n$ is injective for all $n < 0$.

Recall that X_{Nis} has finite cohomological dimension for sheaves of abelian groups. Since \bar{K}^* is acyclic and \bar{K}^n is injective for all $n < 0$, it follows that \bar{K}^0 is an injective sheaf and thus the entire complex \bar{K}^* splits: $\bar{K}^n = \ker(\bar{d}^n) \oplus \ker(\bar{d}^{n+1})$ with the differential $\bar{K}^n \rightarrow \bar{K}^{n+1}$ the projection on $\ker(\bar{d}^{n+1})$ followed by the inclusion. This easily implies that \bar{K}^* is K -injective in $Ch(Sh^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$.

In addition, the exact sequence

$$0 \rightarrow \bar{K}^0 \rightarrow \bar{X}^0 \xrightarrow{\bar{f}^0} \bar{Y}^0 \rightarrow 0$$

splits; let $s : \bar{X}^0 \rightarrow \bar{K}^0$ be a choice of splitting. Taking an injective resolution

$$0 \rightarrow \bar{Y}^0 \xrightarrow{\epsilon} I^0 \xrightarrow{\bar{d}^0} I^1 \rightarrow \dots$$

of \bar{Y}^0 gives the injective resolution

$$0 \rightarrow \bar{X}^0 \xrightarrow{(s, \bar{f}^0 \epsilon)} \bar{K}^0 \oplus I^0 \xrightarrow{\bar{d}^0 \circ \pi_{I^0}} I^1 \rightarrow \dots$$

of \bar{X}^0 . Letting

$$\tilde{Y}^n := \begin{cases} \bar{Y}^n & \text{for } n < 0 \\ I^n & \text{for } n \geq 0, \end{cases}$$

and

$$\tilde{X}^n := \begin{cases} \bar{X}^n & \text{for } n < 0 \\ \bar{K}^0 \oplus I^0 & \text{for } n = 0 \\ I^n & \text{for } n > 0, \end{cases}$$

we can thus extend the map $\bar{f} : \bar{X}^* \rightarrow \bar{Y}^*$ to a fibration $\tilde{f} : \tilde{X}^* \rightarrow \tilde{Y}^*$ with acyclic kernel \bar{K}^* , in other words, \tilde{f} is a fibration and weak equivalence in $Ch(Sh^{\mathbf{Ab}}(X_{\text{Nis}}))$; $\tau^{\leq 0} \tilde{f} = \bar{f}$ by construction.

To finish the proof, let $\bar{K}^* \in Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\text{Nis}}))$ be a finite complex with \bar{K}^n injective for $n < 0$. Splicing in an injective resolution of \bar{K}^0 , we may write $\bar{K}^* = \tau^{\leq 0} K^*$, where K^* is a bounded below complex of injectives. It suffices to show that K^* is K -injective. If A^* is an acyclic complex in $Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\text{Nis}}))$, then as K^* is bounded below, $H^m(\text{Hom}(A^*, K^*)^*) = H^m(\text{Hom}(\tau_{\geq N} A^*, K^*)^*)$ for all $N \ll 0$. Since $\tau_{\geq N} A^*$ is acyclic and bounded below, $\text{Hom}(\tau_{\geq N} A^*, K^*)^*$ is acyclic, and thus \bar{K}^* is K -injective. \square

Proposition 11.4. *Let $j : A \rightarrow X$ be an open immersion in \mathbf{Sm}/k . For each $\phi \in [X, B\pi_1^{\mathbb{A}^1} \mathcal{B}]_{\mathcal{H}_{\text{Nis}}(k)}$, there is a natural bijection*

$$[(X, A), (K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i+1), B\pi_1^{\mathbb{A}^1} \mathcal{B})]_{\mathcal{H}_{\text{Nis}}(k)}^{\phi} \cong H_{\text{Nis}}^{i+1}(X, A, \tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F}^{\phi}).$$

Proof. This result in the absolute case (X, \emptyset) is [26, Theorem B.3.8], and is essentially an application of the Dold-Kan correspondence in the setting of abelian group objects in $\mathbf{Spc}_{\text{Nis}}(X)$. In the case of topological spaces, this result is proven in [31, Proposition 3.6]

Write G for $\pi_1^{\mathbb{A}^1} \mathcal{B}|_{X_{\text{Nis}}}$. We fix a map $\phi : X \rightarrow B\pi_1^{\mathbb{A}^1} \mathcal{B}$ in $\mathcal{H}_{\text{Nis}}(k)$, giving the corresponding element $[\phi] \in H_{\text{Nis}}^1(X, G)$ which we represent by a G -torsor $\mathcal{Y} \rightarrow X$. We write M_{ϕ} for the sheaf $\tilde{\pi}_i^{\mathbb{A}^1} \mathcal{F}^{\phi} := Y \times_G \pi_i^{\mathbb{A}^1} \mathcal{F}|_{X_{\text{Nis}}}$ on X_{Nis} . We note that $\phi^* K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i+1)$ is isomorphic in $\mathcal{H}_{\text{Nis}}(X)$ to the sheaf of Eilenberg-MacLane spaces $K(M_{\phi}, i+1) \in \mathbf{Spc}_{\bullet}(X)$ (see [26, Lemma B.3.7]), and that

$$\phi^*(s) : \phi^* B\pi_1^{\mathbb{A}^1} \mathcal{B} \rightarrow \phi^* K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i+1)$$

goes over to the base-point section $*_X \hookrightarrow K(M_{\phi}, i+1)$ under this identification. This gives us the isomorphism

$$\begin{aligned} [(X, A), (K^{\pi_1^{\mathbb{A}^1} \mathcal{B}}(\pi_i^{\mathbb{A}^1} \mathcal{F}, i+1), B\pi_1^{\mathbb{A}^1} \mathcal{B})]_{\mathcal{H}_{\text{Nis}}(k)}^{\phi} \\ \cong [(X, A), (K(M_{\phi}, i+1), *_X)]_{\mathcal{H}_{\text{Nis}}(X)}. \end{aligned}$$

We work in the model category of (cohomological) complexes of sheaves of abelian groups on X_{Nis} , supported in degrees ≤ 0 , $Ch^{\leq 0}(Sh^{\mathbf{Ab}}(X_{\text{Nis}}))$, with the injective model structure.

Let $M_\phi \rightarrow \mathcal{I}^*$ be an injective resolution of M_ϕ . Under the Dold-Kan correspondence, $K(M_\phi, i+1)$ goes to the complex of sheaves $\tau_{\leq 0}(\mathcal{I}^*[i+1])$ on X_{Nis} , and a fibrant model of the inclusion $*_X \hookrightarrow K(M_\phi, i+1)$ goes to an acyclic fibrant complex \mathcal{F}^* mapping by a fibration to $\tau_{\leq 0}(\mathcal{I}^*[i+1])$,

$$\rho : \mathcal{F}^* \rightarrow \tau_{\leq 0}(\mathcal{I}^*[i+1]).$$

This identifies $[(X, A), (K(M_\phi, i+1), *_X)]_{\mathcal{H}_{\text{Nis}}(X)}$ with

$$H^0(\tau_{\leq 0}(\mathcal{I}^*(X)[i+1]) \times_{\tau_{\leq 0}(\mathcal{I}^*(A)[i+1])} \mathcal{F}^*(A)).$$

Since ρ is a fibration, the map

$$\rho(A) : \mathcal{F}^*(A) \rightarrow \tau_{\leq 0}(\mathcal{I}^*(A)[i+1])$$

is surjective in degrees < 0 and thus the same is true for the projection

$$\tau_{\leq 0}(\mathcal{I}^*(X)[i+1]) \times_{\tau_{\leq 0}(\mathcal{I}^*(A)[i+1])} \mathcal{F}^*(A) \rightarrow \tau_{\leq 0}(\mathcal{I}^*(X)[i+1]).$$

This shows that the evident map

$$\begin{aligned} & \tau_{\leq 0}(\mathcal{I}^*(X)[i+1]) \times_{\tau_{\leq 0}(\mathcal{I}^*(A)[i+1])} \mathcal{F}^*(A) \\ & \rightarrow \text{Cone}(\mathcal{I}^*(X)[i+1] \oplus \mathcal{F}^*(A) \xrightarrow{(\text{res}, \rho)} \mathcal{I}^*(A)[i+1])[-1] \end{aligned}$$

induces an isomorphism on H^i for all $i \leq 0$ and in particular, we have

$$H^0(\tau_{\leq 0}\mathcal{I}^*(X)[i+1] \times_{\tau_{\leq 0}\mathcal{I}^*(A)[i+1]} \mathcal{F}^*(A)) \cong H^{i+1}(X, A, M_\phi).$$

□

Remark 11.5. If the $\pi_1^{\mathbb{A}^1}\mathcal{B}$ action on $\pi_i^{\mathbb{A}^1}\mathcal{F}$ is trivial, then $K^{\pi_1^{\mathbb{A}^1}\mathcal{B}}(\pi_i^{\mathbb{A}^1}\mathcal{F}, i+1) = K(\pi_i^{\mathbb{A}^1}\mathcal{F}, i+1) \times B\pi_1^{\mathbb{A}^1}\mathcal{B}$, and we have \mathbb{A}^1 -homotopy pullback square

$$\begin{array}{ccc} \mathcal{E}^{(i+1)} & \xrightarrow{\quad} & pt \\ \downarrow & & \downarrow s \\ \mathcal{E}^{(i)} & \xrightarrow[k_{i+1}]{} & K(\pi_i^{\mathbb{A}^1}\mathcal{F}, i+1). \end{array}$$

In this case, we have

$$[(X, A), (K(\pi_i^{\mathbb{A}^1}\mathcal{F}, i+1), *)]_{\mathcal{H}_{\text{Nis}}(k)} \cong [X/A, K(\pi_i^{\mathbb{A}^1}\mathcal{F}, i+1)]_{\mathcal{H}_{\bullet, \text{Nis}}(k)}$$

and Proposition 11.4 is just the usual identification

$$[X/A, K(\pi_i^{\mathbb{A}^1}\mathcal{F}, i+1)]_{\mathcal{H}_{\bullet, \text{Nis}}(k)} \cong H_A^{i+1}(X_{\text{Nis}}, \pi_i^{\mathbb{A}^1}\mathcal{F}_{|X_{\text{Nis}}}).$$

Consider the sequence (for $n \geq 2$)

$$(11.1) \quad \text{GL}_n / \text{GL}_{n-1} \rightarrow \text{BGL}'_{n-1} \rightarrow \text{BGL}_n$$

induced by the inclusion $\text{GL}_{n-1} \rightarrow \text{GL}_n$,

$$g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}.$$

Here we take BGL_n to be the colimit presheaf

$$\mathrm{BGL}_n := \mathrm{colim}_N \mathrm{GL}_N / \mathrm{GL}_n \times \mathrm{GL}_{N-n}$$

and use the model

$$\mathrm{colim}_N \mathrm{GL}_N / \mathrm{GL}_{n-1} \times \mathrm{GL}_{N-n}$$

for BGL'_{n-1} . By [27, Proposition 4.3.7] these models agree with the model of BGL_n and BGL_{n-1} formed by using the standard simplicial schemes, and by [6, Theorem 1], the sequence (11.1) is an \mathbb{A}^1 homotopy fiber sequence.

It will be more convenient to replace (11.1) with the sequence

$$(11.2) \quad \mathbb{A}^n \setminus \{0\} \rightarrow \mathrm{BGL}_{n-1} \rightarrow \mathrm{BGL}_n$$

where we replace GL_{n-1} with the Euclidean group

$$\mathrm{Eucl}_{n-1} := \left\{ \begin{pmatrix} 1 & v_1 & \cdots & v_{n-1} \\ 0 & & & \\ \vdots & & g & \\ 0 & & & \end{pmatrix}, g \in \mathrm{GL}_{n-1} \right\}$$

and replace BGL'_{n-1} with

$$\mathrm{BGL}_{n-1} := \mathrm{colim}_N \mathrm{GL}_N / \mathrm{Eucl}_{n-1} \times \mathrm{GL}_{N-n}.$$

Since $\mathrm{GL}_N / \mathrm{GL}_{n-1} \times \mathrm{GL}_{N-n} \rightarrow \mathrm{GL}_N / \mathrm{Eucl}_{n-1} \times \mathrm{GL}_{N-n}$ is an affine space bundle, as is the induced map $\mathrm{GL}_n / \mathrm{GL}_{n-1} \rightarrow \mathbb{A}^n \setminus \{0\}$, the sequence (11.2) is isomorphic to the sequence (11.1) in $\mathcal{H}(k)$; in particular, (11.2) is also an \mathbb{A}^1 homotopy fiber sequence.

For notational simplicity, we set

$$\begin{aligned} \mathrm{Gr}_{aff}(n, N) &:= \mathrm{GL}_N / \mathrm{GL}_n \times \mathrm{GL}_{N-n}, \\ \mathrm{Gr}_{aff}(n-1, N) &:= \mathrm{GL}_N / \mathrm{Eucl}_{n-1} \times \mathrm{GL}_{N-n}. \end{aligned}$$

Let $p_{n,N} : E_{n,N} \rightarrow \mathrm{Gr}_{aff}(n, N)$ be the “universal” bundle

$$E_{n,N} := \mathrm{GL}_N \times_{\mathrm{GL}_n \times \mathrm{GL}_{N-n}} \mathbb{A}^n,$$

with $\mathrm{GL}_n \times \mathrm{GL}_{N-n}$ acting on \mathbb{A}^n through the standard left action of GL_n on \mathbb{A}^n . Let $p_{n-1,N} : E_{n-1,N} \rightarrow \mathrm{Gr}_{aff}(n-1, N)$ be the similarly defined rank $n-1$ bundle, using the action of Eucl_{n-1} on \mathbb{A}^{n-1} through its quotient $\mathrm{Eucl}_{n-1} \rightarrow \mathrm{GL}_{n-1}$.

Asok [2, Lemma 3.10] has shown that $\pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n) = \mathbb{G}_m$, with classifying map $\mathrm{BGL}_n \rightarrow B\mathbb{G}_m$ induced by the determinant $\mathrm{GL}_n \rightarrow \mathbb{G}_m$. Let

$$\tilde{\mathrm{Eucl}}_{n-1} := \left\{ \begin{pmatrix} 1 & v_1 & \cdots & v_{n-1} \\ 0 & & & \\ \vdots & & g & \\ 0 & & & \end{pmatrix}, g \in \mathrm{SL}_{n-1} \right\}$$

and define

$$\begin{aligned}\tilde{\mathrm{Gr}}_{aff}(n, N) &:= \mathrm{GL}_N / \mathrm{SL}_n \times \mathrm{GL}_{N-n}, \\ \tilde{\mathrm{Gr}}_{aff}(n-1, N) &:= \mathrm{GL}_N / \mathrm{Eucl}_{n-1} \times \mathrm{GL}_{N-n}.\end{aligned}$$

Let

$$\mathrm{BSL}_n := \mathrm{colim}_N \tilde{\mathrm{Gr}}_{aff}(n, N); \quad \mathrm{BSL}_{n-1} := \mathrm{colim}_N \tilde{\mathrm{Gr}}_{aff}(n-1, N)$$

As above, BSL_{n-1} is isomorphic in $\mathcal{H}(k)$ to

$$\mathrm{BSL}'_{n-1} := \mathrm{colim}_N \mathrm{GL}_N / \mathrm{SL}_{n-1} \times \mathrm{GL}_{N-n}.$$

It follows from [27, Lemma 1.8, Proposition 2.6, Remark 2.7] and the fact [33, 4.4(b)] that $H_{\mathrm{et}}^1(X, \mathrm{SL}_n) = H_{\mathrm{Nis}}^1(X, \mathrm{SL}_n)$ for all $X \in \mathbf{Sm}/k$ that BSL_n and BSL_{n-1} are isomorphic in $\mathcal{H}(k)$ to the respective simplicial classifying spaces. The sequence

$$\mathrm{GL}_n / \mathrm{SL}_n \rightarrow \mathrm{BSL}_n \rightarrow \mathrm{BGL}_n$$

is an \mathbb{A}^1 homotopy fiber sequence, using [6, Theorem 1] and [7, Theorem 4.1.1]; the identification of $\mathrm{GL}_n / \mathrm{SL}_n$ with \mathbb{G}_m by the determinant shows that $\mathrm{BSL}_n \rightarrow \mathrm{BGL}_n$ is the universal cover of BGL_n , in particular, BSL_n and BSL_{n-1} are \mathbb{A}^1 -simply connected.

Using [7, Theorem 4.1.1], we see that $\mathbb{A}^n \setminus \{0\} \rightarrow \mathrm{BSL}_{n-1} \rightarrow \mathrm{BSL}_n$ is an \mathbb{A}^1 homotopy fiber sequence. The evident maps give the commutative diagram of \mathbb{A}^1 -homotopy fiber sequences

$$(11.3) \quad \begin{array}{ccccc} \mathbb{A}^n \setminus \{0\} & \longrightarrow & \mathrm{BSL}_{n-1} & \longrightarrow & \mathrm{BSL}_n \\ \parallel & & \downarrow & & \downarrow \\ \mathbb{A}^n \setminus \{0\} & \longrightarrow & \mathrm{BGL}_{n-1} & \longrightarrow & \mathrm{BGL}_n \end{array}$$

We have vector bundles $\tilde{p}_{n,N} : \tilde{E}_{n,N} \rightarrow \tilde{\mathrm{Gr}}_{aff}(n, N)$, $\tilde{p}_{n-1,N} : \tilde{E}_{n-1,N} \rightarrow \tilde{\mathrm{Gr}}_{aff}(n-1, N)$ defined as above; these both come with canonical trivializations of their respective determinant bundles.

Let $\mathrm{BGL}_{n-1}^{\mathrm{fib}} \rightarrow \mathrm{BGL}_n^{\mathrm{fib}}$ be a fibration in $\mathbf{Spc}_{\mathrm{Nis}}(k)$, with $\mathrm{BGL}_{n-1}^{\mathrm{fib}}$ and $\mathrm{BGL}_n^{\mathrm{fib}}$ \mathbb{A}^1 fibrant models of BGL_{n-1} and BGL_n , respectively, modeling the map $\mathrm{BGL}_{n-1} \rightarrow \mathrm{BGL}_n$.

For $n \geq 2$, $\mathbb{A}^n \setminus \{0\}$ is \mathbb{A}^1 -connected and Morel's connectedness theorem [26, 6.38] tells us that

$$\pi_k^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \cong \pi_k^{\mathbb{A}^1}(S^{n-1} \wedge \mathbb{G}_m^{\wedge n}) = 0$$

for $1 \leq k \leq n-2$. Furthermore, Morel [26, Theorem 6.40] has defined an isomorphism of Nisnevich sheaves

$$\pi_{n-1}^{\mathbb{A}^1}(S^{n-1} \wedge \mathbb{G}_m^{\wedge n}) \cong \mathcal{K}_n^{MW}.$$

In particular, the fibration $\mathrm{BGL}_{n-1}^{\mathrm{fib}} \rightarrow \mathrm{BGL}_n^{\mathrm{fib}}$ is $n-2$ connected for $n \geq 2$ and in case $n=2$, the fiber has abelian $\pi_1^{\mathbb{A}^1}$.

Referring to the fiber sequence (11.2), Asok-Fasel [3, §6.2] show that the $\pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n)$ -action on $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \cong \mathcal{K}_n^{MW}$ is the canonical one: for $x \in X \in \mathbf{Sm}/k$, a unit $u \in \mathbb{G}_m(\mathcal{O}_{X,x})$ acts on $\mathcal{K}_n^{MW}(\mathcal{O}_{X,x})$ by multiplication by $\langle u \rangle$.

The standard action of \mathbb{G}_m on \mathbb{A}^1 gives us the universal line bundle

$$L_{\mathrm{univ}} \rightarrow B\mathbb{G}_m,$$

$L_{\mathrm{univ}} := E\mathbb{G}_m \times_{\mathbb{G}_m} \mathbb{A}^1$. The action of \mathbb{G}_m on $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \cong \mathcal{K}_n^{MW}$ gives us the bundle of abelian groups $\mathcal{K}_n^{MW}(\det^{-1} L_{\mathrm{univ}}) := E\mathbb{G}_m \times_{\mathbb{G}_m} \mathcal{K}_n^{MW}$ on $B\mathbb{G}_m$ and the corresponding bundle of Eilenberg-MacLane spaces. We thus have a canonical isomorphism of $K^{\mathbb{G}_m}(\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}), n) \rightarrow B\mathbb{G}_m$ with

$$\pi : K(\mathcal{K}_n^{MW}(\det^{-1} L_{\mathrm{univ}}), n) \rightarrow B\mathbb{G}_m.$$

If we have an $X \in \mathbf{Sm}/k$ and a map $\phi : X \rightarrow B\mathbb{G}_m$ in $\mathcal{H}(k)$, we have the corresponding line bundle $L_\phi \rightarrow X$ with $L_\phi \setminus 0_X \rightarrow X$ the \mathbb{G}_m -torsor corresponding to ϕ . Then $K(\mathcal{K}_n^{MW}(\det^{-1} L_{\mathrm{univ}}), n) \times_{B\mathbb{G}_m} X \rightarrow X$ is the bundle of Eilenberg-MacLane spaces $K(\mathcal{K}_n^{MW}(\det^{-1} L_\phi), n) \rightarrow X$. Similarly, we have

$$\tilde{\pi}_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})^\phi = \mathcal{K}_n^{MW}(\det^{-1} L_\phi).$$

We may apply Theorem 11.2 to the map $\mathrm{BGL}_{n-1}^{\mathrm{fib}} \rightarrow \mathrm{BGL}_n^{\mathrm{fib}}$. We have $\mathrm{BGL}_{n-1}^{(k)} = \mathrm{BGL}_n^{\mathrm{fib}}$ for $0 \leq k < n$ and we have the Nisnevich local homotopy cartesian square

$$(11.4) \quad \begin{array}{ccc} \mathrm{BGL}_{n-1}^{(n)} & \xrightarrow{\quad} & B\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathrm{BGL}_n^{\mathrm{fib}} & \xrightarrow[k_n]{} & K(\mathcal{K}_n^{MW}(\det^{-1} L_{\mathrm{univ}}), n). \end{array}$$

For $X \in \mathbf{Sm}/k$, each rank n vector bundle $E \rightarrow X$ gives rise to a well-defined classifying morphism $X \rightarrow \mathrm{BGL}_n$ in $\mathcal{H}(k)$, and thus a well-defined map $\phi_E : X \rightarrow \mathrm{BGL}_n^{\mathrm{fib}}$ in $\mathcal{H}_{\mathrm{Nis}}(k)$. This gives the map

$$k_n \circ \phi_E : X \rightarrow K(\mathcal{K}_n^{MW}(\det^{-1} L_{\mathrm{univ}}), n)$$

in $\mathcal{H}_{\mathrm{Nis}}(k)$, which by Proposition 11.4 yields an element

$$e_{ob}(E) \in H_{\mathrm{Nis}}^n(X, \mathcal{K}_n^{MW}(\det^{-1} E)) = H_{\mathrm{Zar}}^n(X, \mathcal{K}_n^{MW}(\det^{-1} E)).$$

Proposition 11.6. *For each $n \geq 2$ there is a unit $\gamma_n \in \mathrm{GW}(k)^\times$ such that, for each rank n vector bundle $E \rightarrow X$ on some smooth $X \in \mathbf{Sm}/k$, we have*

$$e_{ob}(E) = \gamma_n \cdot e(E).$$

Proof. As Milnor-Witt cohomology is homotopy invariant, we may replace X with a Jouanolou cover and assume from the start that X is affine.

Each rank n vector bundle $V \rightarrow X$ on the affine X is classified by morphism $\phi_{V,N} : X \rightarrow \mathrm{Gr}_{aff}(n, N)$ for $N \gg 0$. Thus, it suffices to prove the result for universal bundle $p_{n,N} : E_{n,N} \rightarrow \mathrm{Gr}_{aff}(n, N)$.

We have the isomorphism

$$q_n : \mathrm{Gr}_{aff}(n-1, N) \rightarrow E_{n,N} \setminus 0_{\mathrm{Gr}(n,N)}$$

induced by sending a matrix $g \in \mathrm{GL}_N$ to the pair $(g, e_1) \in \mathrm{GL}_N \times \mathbb{A}^n$, where e_1 is the first basis vector ${}^t(1, 0, \dots, 0)$. This gives the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_{aff}(n-1, N) & \xrightarrow[\sim]{q_n} & E_{n,N} \setminus 0_{\mathrm{Gr}(n,N)} \\ & \searrow \rho_n & \downarrow j \\ & & E_{n,N} \\ & & \downarrow p_{n,N} \\ & & \mathrm{Gr}_{aff}(n, N) \end{array}$$

where ρ_n is the evident map. This gives us the diagram in $\mathbf{Spc}(k)$

$$\begin{array}{ccccccc} \mathrm{Gr}_{aff}(n-1, N) & \xrightarrow{\phi_{E_{n-1,N}}} & \mathrm{BGL}_{n-1}^{\mathrm{fib}} & \longrightarrow & \mathrm{BGL}_{n-1}^{(n)} & \longrightarrow & B\mathbb{G}_m \\ \downarrow q_n \wr & \searrow \rho_n & \downarrow & \searrow & \downarrow & & \downarrow \\ E_{n,N} \setminus 0_{\mathrm{Gr}_{aff}(n,N)} & & & & & & \\ \downarrow j & & & & & & \\ E_{n,N} & \xrightarrow{p_{n,N}} & \mathrm{Gr}_{aff}(n, N) & \xrightarrow{\phi_{E_{n,N}}} & \mathrm{BGL}_n^{\mathrm{fib}} & \xrightarrow{k_n} & K(\mathcal{K}_n^{MW}(\det^{-1}), n) \end{array}$$

commuting in $\mathcal{H}_{\mathrm{Nis}}(k)$. By Proposition 11.4, this diagram gives rise to a cohomology class $\mathrm{th}_{ob}(E_{n,N}) \in H_{s_0(\mathrm{Gr}_{aff}(n,N))}^n(E_{n,N}, \mathcal{K}_n^{MW}(\det^{-1} E_{n,N}))$ with

$$s_0^* \mathrm{th}_{ob}(E_{n,N}) = e_{ob}(E_{n,N}),$$

where s_0 is the 0-section.

Replace GL_n with SL_n . This gives us the commutative diagram

$$(11.6) \quad \begin{array}{ccccc} \tilde{\mathrm{Gr}}_{aff}(n-1, N) & \longrightarrow & \tilde{\mathrm{Gr}}_{aff}(n, N) & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & \mathrm{BSL}_{n-1} & \longrightarrow & \mathrm{BSL}_n & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathrm{Gr}_{aff}(n-1, N) & \longrightarrow & \mathrm{Gr}_{aff}(n, N) & & \\ & \downarrow & \downarrow & \downarrow & \\ & \mathrm{BGL}_{n-1} & \longrightarrow & \mathrm{BGL}_n & \end{array}$$

Let $\tilde{E}_{n,N} \rightarrow \tilde{\mathrm{Gr}}_{aff}(n, N)$ be the universal bundle. As above, we have a commutative diagram

$$\begin{array}{ccc}
 \tilde{\mathrm{Gr}}_{aff}(n-1, N) & \xrightarrow[\sim]{\tilde{q}_n} & \tilde{E}_{n,N} \setminus 0_{\mathrm{Gr}(n,N)} \\
 & \searrow \tilde{\rho}_n & \downarrow \tilde{j} \\
 & & \tilde{E}_{n,N} \\
 & & \downarrow p_{n,N} \\
 & & \tilde{\mathrm{Gr}}_{aff}(n, N)
 \end{array}$$

Repeating the Moore-Postnikov construction for the fibration sequence

$$\mathbb{A}^n \setminus \{0\} \rightarrow \mathrm{BSL}_{n-1}^{\mathrm{fib}} \rightarrow \mathrm{BSL}_n^{\mathrm{fib}}$$

and noting that BSL_n is \mathbb{A}^1 -simply connected, we have the diagram in $\mathbf{Spc}(k)$

$$\begin{array}{ccccccc}
 \tilde{\mathrm{Gr}}_{aff}(n-1, N) & \xrightarrow{\phi_{\tilde{E}_{n-1,N}}} & \mathrm{BSL}_{n-1}^{\mathrm{fib}} & \longrightarrow & \mathrm{BSL}_{n-1}^{(n)} & \longrightarrow & pt \\
 \downarrow q_n \wr & \searrow \tilde{\rho}_n & \downarrow & & \downarrow & & \downarrow \\
 \tilde{E}_{n,N} \setminus 0_{\tilde{\mathrm{Gr}}_{aff}(n,N)} & & \tilde{E}_{n,N} & \xrightarrow{\tilde{p}_{n,N}} & \tilde{\mathrm{Gr}}_{aff}(n-1, N) & \xrightarrow{\phi_{\tilde{E}_{n-1,N}}} & \mathrm{BSL}_n^{\mathrm{fib}} \xrightarrow{\tilde{k}_n} K(\mathcal{K}_n^{MW}, n+1) \\
 \downarrow \tilde{j} & & & & & & \\
 \tilde{E}_{n,N} & & & & & &
 \end{array}$$

commuting in $\mathcal{H}_{\mathrm{Nis}}(k)$. We have as well a map of diagram (11.7) to the diagram (11.5), extending the comparison maps in (11.3) and (11.6).

As before, this gives us the cohomology class

$$\mathrm{th}_{ob}(\tilde{E}_{n,N}) \in H_{s_0(\tilde{\mathrm{Gr}}_{aff}(n,N))}^n(\tilde{\mathrm{Gr}}_{aff}(n, N), \mathcal{K}_n^{MW})$$

with $s_0^* \mathrm{th}_{ob}(\tilde{E}_{n,N}) = e_{ob}(\tilde{E}_{n,N})$.

By [4, Theorem 5.3.2], there is a universal element $\gamma_n \in \mathrm{GW}(k)^\times$ with

$$e_{ob}(\tilde{E}_{n,N}) = \gamma_n \cdot e(\tilde{E}_{n,N}).$$

The proof of [4, Theorem 5.3.2] is in fact a comparison of $\mathrm{th}_{ob}(\tilde{E}_{n,N})$ with $\mathrm{th}(\tilde{E}_{n,N})$, showing that

$$(11.8) \quad \mathrm{th}_{ob}(\tilde{E}_{n,N}) = \gamma_n \cdot \mathrm{th}(\tilde{E}_{n,N}).$$

Take an affine open cover $\mathcal{U} = \{U_\alpha\}$ of $\mathrm{Gr}_{aff}(n, N)$ trivializing the line bundle $\det E_{n,N}$. For each α let $j_\alpha : U_\alpha \rightarrow \mathrm{Gr}_{aff}(n, N)$ be the inclusion and choose an isomorphism $\lambda_\alpha : j_\alpha^* \det E_{n,N} \cong \mathcal{O}_{U_\alpha}$. The isomorphism λ_α defines a section

$$\sigma_\alpha : U_\alpha \rightarrow \tilde{\mathrm{Gr}}_{aff}(n, N)$$

to the projection $\tilde{\mathrm{Gr}}_{aff}(n, N) \rightarrow \mathrm{Gr}_{aff}(n, N)$ and an isomorphism $\psi_\alpha : \sigma_\alpha^* \tilde{E}_{n,N} \rightarrow j_\alpha^* E_{n,N}$. Thus

$$(\sigma_\alpha, \lambda_\alpha)^*(\mathrm{th}_{ob}(\tilde{E}_{n,N})) = j_\alpha^*(\mathrm{th}_{ob}(E_{n,N}))$$

in $H_{0_{U_\alpha}}^n(E_{n,N}|_{U_\alpha}, \mathcal{K}_n^{MW}(\det^{-1} E_{n,N}))$. The identity (11.8) and the functoriality of $\mathrm{th}(-)$ shows that

$$j_\alpha^*(\mathrm{th}_{ob}(E_{n,N})) = \gamma_n \cdot j_\alpha^* \mathrm{th}(\tilde{E}_{n,N})$$

in $H_{0_{U_\alpha}}^n(E_{n,N}|_{U_\alpha}, \mathcal{K}_n^{MW}(\det^{-1} E_{n,N}))$ for each α .

We have the Thom isomorphism

$$\begin{aligned} H^0(\mathrm{Gr}_{aff}(n, N), \mathcal{K}_0^{MW}) \\ \xrightarrow{\mathrm{th}(E_{n,N}) \cup -} H_{0_{\mathrm{Gr}_{aff}(n, N)}}^n(E_{n,N}, \mathcal{K}_n^{MW}(\det^{-1} E_{n,N})); \end{aligned}$$

let $x_{ob} \in H^0(\mathrm{Gr}_{aff}(n, N), \mathcal{K}_0^{MW})$ be the element with $\mathrm{th}(E_{n,N}) \cup x_{ob} = \mathrm{th}_{ob}(E_{n,N})$. The identity (11.8) shows that $j_\alpha^* x_{ob} = \gamma_n$ for all α and thus $x_{ob} = \gamma_n$. This shows that

$$\mathrm{th}_{ob}(E_{n,N}) = \gamma_n \cdot \mathrm{th}(E_{n,N}).$$

As the obstruction Euler class $e_{ob}(E_{n,N})$ is given by

$$e_{ob}(E_{n,N}) := s_0^*(\mathrm{th}_{ob}(E_{n,N}))$$

and $e(E_{n,N}) = s_0^*(\mathrm{th}(E_{n,N}))$, we have $e_{ob}(E_{n,N}) = \gamma_n \cdot e(E_{n,N})$, completing the proof. \square

Remark 11.7. The rank two case of Theorem 11.1 follows from the isomorphism $V^\vee \cong V \otimes \det^{-1} V$ for a rank two vector bundle V , and Remark 10.7(2), which gives

$$e(V^\vee) = e(V \otimes \det^{-1} V) = e(V) + e(\det^{-1} V) \cup e(\det^{-1} V \otimes \det V) = e(V).$$

We proceed to prove Theorem 11.1. If a rank n vector bundle E on X is classified by a map $f : X \rightarrow \mathrm{BGL}_n$, then the dual E^\vee is classified by $B\alpha_n \circ f$, where $B\alpha_n : \mathrm{BGL}_n \rightarrow \mathrm{BGL}_n$ is the automorphism induced by the automorphism of group-schemes $\alpha_n : \mathrm{GL}_n \rightarrow \mathrm{GL}_n$, $\alpha_n(g) = {}^t g^{-1}$.

Thus, for rank $n \geq 2$, the theorem follows from

Proposition 11.8. *Via the \mathbb{A}^1 -weak equivalence $\mathrm{GL}_n / \mathrm{GL}_{n-1} \rightarrow \mathbb{A}^n \setminus \{0\}$, and the isomorphism $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}, e_1) \cong \mathcal{K}_n^{MW}$, the automorphism α_n induces the map $\times(-\langle -1 \rangle)^n$ on \mathcal{K}_n^{MW} .*

Indeed, Asok-Fasel [4, Proposition 3.4.1] compute action of $\pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n) \cong \mathbb{G}_m$ on $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}, e_1) \cong \mathcal{K}_n^{MW}$ as $t \cdot x = \langle t \rangle \cdot x$ for $t \in \mathbb{G}_m(U)$ and $x \in \mathcal{K}_n^{MW}(U)$. As $\alpha_{n*} : \pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n) \rightarrow \pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n)$ is the map $t \mapsto t^{-1}$ on \mathbb{G}_m , the fact that $\langle t \rangle = \langle t^{-1} \rangle$ implies that there is a canonical isomorphism

$$\alpha_n^* K_{\mathrm{BGL}_n}(\mathcal{K}_n^{MW}(\det^{-1} \mathcal{E}_n), n) \cong K(\mathcal{K}_n^{MW}(\det^{-1} \mathcal{E}_n), n).$$

As $\times(-\langle -1 \rangle)^n$ commutes with the $\pi_1^{\mathbb{A}^1}(\mathrm{BGL}_n)$ -action, this together with Proposition 11.8 shows that $k_{n-1} \circ \alpha_n = (-\langle -1 \rangle)^n \times k_{n-1}$.

To prove the proposition, let $Q_{2n-1} \subset \mathbb{A}^{2n}$ be the affine quadric defined by

$$\sum_{i=1}^n x_i y_i = 1.$$

We have the projection $\pi : Q_{2n-1} \rightarrow \mathbb{A}^n \setminus \{0\}$, $\pi(x, y) = x$, realizing Q_{2n-1} as an \mathbb{A}^{n-1} bundle over $\mathbb{A}^n \setminus \{0\}$. We have the map $\tilde{\rho} : \mathrm{GL}_n \rightarrow Q_{2n-1}$ with

$$\tilde{\rho}((g_{ij})) := (g_{11}, \dots, g_{n1}, g^{11}, \dots, g^{1i}, \dots, g^{1n}),$$

where g^{ij} is the ij th entry in the cofactor matrix ${}^t(g_{ij})^{-1}$ to (g_{ij}) . As

$$\tilde{\rho}\left(g \cdot \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}\right) = \tilde{\rho}(g),$$

$\tilde{\rho}$ induces the map

$$\rho : \mathrm{GL}_n / \mathrm{GL}_{n-1} \rightarrow Q_{2n-1}$$

with $\pi\rho(g) = g \cdot e_1$. It follows easily from this that ρ is an isomorphism of affine schemes over k and thus ρ induces an isomorphism

$$(11.9) \quad (Q_{2n-1}, *) \cong S^{n-1} \wedge \mathbb{G}_m^{\wedge n}$$

in $\mathcal{H}_\bullet(k)$, for any choice of base-point $* \in Q_{2n-1}$; for concreteness, we use $* = (1, 0, \dots, 0; 1, 0, \dots, 0)$.

In addition the automorphism α_n induces the automorphism

$$\beta_n : Q_{2n-1} \rightarrow Q_{2n-1}; \quad \beta_n(x, y) = (y, x).$$

The proposition will follow from

Proposition 11.9. *Suppose $n \geq 2$. Via the isomorphism (11.9), the automorphism β_n induces the map $\times(-\langle -1 \rangle)^n$ on \mathcal{K}_n^{MW} .*

The proof uses the next two lemmas.

Lemma 11.10. *Choose integers i, j with $1 \leq i < j \leq n$ and let*

$$\tau_{i,j} : Q_{2n-1} \rightarrow Q_{2n-1}$$

be the map

$$\begin{aligned} \tau_{i,j}(x_1, \dots, x_i, \dots, x_j, \dots, x_n; y_1, \dots, y_i, \dots, y_j, \dots, y_n) \\ = (x_1, \dots, y_i, \dots, y_j, \dots, x_n; y_1, \dots, x_i, \dots, x_j, \dots, y_n) \end{aligned}$$

that is, exchange x_i for y_i , exchange x_j for y_j and leaving x_ℓ, y_ℓ unchanged for $\ell \notin \{i, j\}$. Then $\tau_{i,j}$ is \mathbb{A}^1 -homotopic to the identity on Q_{2n-1} .

Proof. Send the tuple (x_i, x_j, y_i, y_j) to the matrix

$$m = m(x_i, x_j, y_i, y_j) := \begin{pmatrix} x_i & -y_j \\ x_j & y_i \end{pmatrix}$$

Take $g \in \mathrm{SL}_2$ and let (x'_i, x'_j, y'_i, y'_j) be the tuple with

$$m(x'_i, x'_j, y'_i, y'_j) = g \cdot m(x_i, x_j, y_i, y_j)$$

Since $\det m(x_i, x_j, y_i, y_j) = x_i y_i + x_j y_j$, we see that, given a point $(x, y) \in Q$, the point (x', y') formed by replacing (x_i, x_j, y_i, y_j) with (x'_i, x'_j, y'_i, y'_j) and leaving all other x_ℓ, y_ℓ unchanged is also in Q_{2n-1} , and we thus have an action

$$\mu_{i,j}^\ell : \mathrm{SL}_2 \times_k Q_{2n-1} \rightarrow Q_{2n-1}.$$

Similarly, right multiplication gives us the (right) action

$$\mu_{i,j}^r : Q_{2n-1} \times_k \mathrm{SL}_2 \rightarrow Q_{2n-1}.$$

Since $\mathrm{SL}_2(k)$ is generated by elementary matrices, it follows that for $g \in \mathrm{SL}_2(k)$, the automorphisms $\mu_{i,j}^\ell(g, -) : Q_{2n-1} \rightarrow Q_{2n-1}$ and $\mu_{i,j}^r(-, g) : Q_{2n-1} \rightarrow Q_{2n-1}$ are \mathbb{A}^1 -homotopic to the identity. Since

$$m(y_i, y_j, x_i, x_j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot m(x_i, x_j, y_i, y_j) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the result follows. \square

This lemma handles the case of even n and reduces the case of odd n to proving the following result.

Lemma 11.11. *Let $n \geq 3$ be odd and let $\sigma_n : Q_{2n-1} \rightarrow Q_{2n-1}$ be the automorphism*

$$\sigma_n(x_1, \dots, x_n, y_1, \dots, y_n) = (x_1, \dots, x_{n-1}, y_n; y_1, \dots, y_{n-1}, x_n).$$

*Then via the isomorphism $(Q_{2n-1}, *) \cong \Sigma_{S^1}^{n-1} \mathbb{G}_m^{\wedge n}$ in $\mathcal{H}_\bullet(k)$, ρ_n gives the automorphism $\Sigma^{n-1} \tilde{\sigma}_n$, where $\tilde{\sigma}_n$ is the automorphism of $\mathbb{G}_m^{\wedge n}$*

$$\tilde{\sigma}_n(t_1 \wedge \dots \wedge t_n) = t_1 \wedge \dots \wedge t_{n-1} \wedge t_n^{-1}.$$

That this does in fact prove Proposition 11.8 follows from the identity in $K_1^{MW}(\mathbb{G}_m)$

$$[t^{-1}] = (-\langle -1 \rangle)[t];$$

see [26, Lemma 3.14].

Proof of Lemma 11.11. We begin by defining a homotopy inverse to the map $Q_{2n-1} \rightarrow \mathbb{A}^n \setminus \{0\}$ in $\mathbf{Spc}_\bullet(k)$.

For this, we recall a construction of an “ \mathbb{A}^1 geometric realization” of a simplicial object \mathcal{X}_\bullet in $\mathbf{Spc}(k)$. This is the quotient presheaf

$$|\mathcal{X}_\bullet| := \coprod_m \mathcal{X}_m \times \Delta_k^m / \sim$$

where $m \mapsto \Delta_k^m$ is the standard cosimplicial scheme of algebraic m -simplices and \sim is the usual equivalence relation

$$(x, \Delta(g)(t)) \sim (\mathcal{X}(g)(x), t)$$

on T -valued points $(x, t) \in \mathcal{X}_n \times \Delta_k^m$, with $g : [m] \rightarrow [n]$ in **Ord**. If $\check{\mathcal{U}}_\bullet$ is the Čech simplicial scheme associated to a finite Zariski open cover $\mathcal{U} := \{U_i\}_{i \in I}$ of a smooth k -scheme Y , then the collection of morphisms

$$(U_{i_0} \cap \dots \cap U_{i_m}) \times \Delta_k^m \xrightarrow{p_1} U_{i_0} \cap \dots \cap U_{i_m} \xrightarrow{j_{i_*}} Y,$$

where j_{i_*} is the inclusion, define an \mathbb{A}^1 -weak equivalence

$$\pi_{\mathcal{U}} : |\check{\mathcal{U}}_\bullet| \rightarrow Y.$$

Furthermore, suppose we have a morphism $f : Y \rightarrow X$, an affine space bundle $q : V \rightarrow X$, and liftings $f_i : U_i \rightarrow V$ of $f|_{U_i}$ for some Zariski open cover $\mathcal{U} := \{U_i\}_{i \in I}$ of Y . By an affine space bundle, we mean a torsor (in the Zariski topology) for a vector bundle $E \rightarrow X$, the latter considered as a group-scheme over X . For an index $i_* := (i_0, \dots, i_m) \in I^{m+1}$, let $U_{i_*} = \cap_{j=0}^m U_{i_j}$ and define the map

$$f_{i_*} : U_{i_*} \times \Delta_k^m \rightarrow V$$

(on T -valued points) by

$$f_{i_*}(u; t_0, \dots, t_m) = \sum_{j=0}^m t_j f_{i_j}(u).$$

The fact that $q(f_j(u)) = q(f_{j'}(u))$ for all j, j' , that $\sum_{j=0}^m t_j = 1$ and that the fiber $V_{q(f_j(u))}$ is a torsor for $E_{q(f_j(u))}$ readily implies that the sum used in this definition makes sense and the formula for $f_{i_*}(u; t_0, \dots, t_m)$ does give a well-defined morphism f_{i_*} . One easily checks that the collection of maps f_{i_*} fit together to give a well-defined morphism in **Spc**(k)

$$\tilde{f}_{\mathcal{U}} : |\check{\mathcal{U}}_\bullet| \rightarrow V$$

lifting $f \circ \pi_{\mathcal{U}}$.

We apply these constructions to the open cover $\mathcal{U} := \{U_1, \dots, U_n\}$ of $\mathbb{A}^n \setminus \{0\}$, with $U_i = \{(x_1, \dots, x_n) \mid x_i \neq 0\}$ and the morphism $q : Q_{2n-1} \rightarrow \mathbb{A}^n \setminus \{0\}$, which makes Q_{2n-1} an \mathbb{A}^{n-1} -bundle over $\mathbb{A}^n \setminus \{0\}$. Over U_i , we have the section $s_i : U_i \rightarrow Q_{2n-1}$ to q , defined by

$$s_i(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0, x_i^{-1}, 0, \dots, 0)$$

that is, $y_j = 0$ for $j \neq i$ and $y_i = x_i^{-1}$. This gives us the lifting

$$s := \tilde{\text{Id}}_{\mathcal{U}} : |\check{\mathcal{U}}_\bullet| \rightarrow Q_{2n-1}$$

of $\pi_{\mathcal{U}} : |\check{\mathcal{U}}_\bullet| \rightarrow \mathbb{A}^n \setminus \{0\}$.

We consider the composition

$$|\check{\mathcal{U}}_\bullet| \xrightarrow{s} Q_{2n-1} \xrightarrow{\sigma_n} Q_{2n-1} \xrightarrow{q} \mathbb{A}^n \setminus \{0\},$$

which we denote by

$$\check{\sigma}_n : |\check{\mathcal{U}}_\bullet| \rightarrow \mathbb{A}^n \setminus \{0\}.$$

For $i_* = (i_0, \dots, i_m)$, this map on $U_{i_*} \times \Delta_k^m$ is given by

$$((x_1, \dots, x_n), (t_0, \dots, t_m)) \mapsto \begin{cases} (x_1, \dots, x_{n-1}, 0) & \text{if } i_m < n \\ (x_1, \dots, x_{n-1}, t_m x_n^{-1}) & \text{if } i_m = n. \end{cases}$$

Thus, the composition

$$\check{\sigma}_n \circ \pi_{\mathcal{U}}^{-1} : (\mathbb{A}^n \setminus \{0\}, 1^n) \rightarrow (\mathbb{A}^n \setminus \{0\}, 1^{n-1} \times \mathbb{A}^1)$$

in $\mathcal{H}_\bullet(k)$ can also be expressed using the cover \mathcal{V} of $\mathbb{A}^n \setminus \{0\}$ by the two open subsets $V_0 := (\mathbb{A}^{n-1} \setminus \{0\}) \times \mathbb{A}^1$ and $V_1 := \mathbb{A}^{n-1} \times (\mathbb{A}^1 \setminus \{0\})$. The \mathbb{A}^1 -geometric realization $|\check{\mathcal{V}}_\bullet|$ is equivalent to the glued space

$$V_* := V_0 \times 0 \amalg V_1 \times 1 \amalg V_{01} \times \mathbb{A}^1 / \sim,$$

with $V_{01} = V_0 \cap V_1$ and \sim identifying $(v, 1) \in V_{01} \times \mathbb{A}^1$ with $v \times 1 \in V_1 \times 1$ and $(v, 0) \in V_{01} \times \mathbb{A}^1$ with $v \times 0 \in V_0 \times 0$. The map $\check{\sigma}_n \circ \pi_{\mathcal{U}}^{-1}$ (in $\mathcal{H}_\bullet(k)$) is thus induced by the map $V_* \rightarrow \mathbb{A}^n \setminus \{0\}$ given by

$$\begin{aligned} (x_1, \dots, x_{n-1}, x_n) &\mapsto (x_1, \dots, x_{n-1}, 0) \text{ on } V_0 \\ (x_1, \dots, x_{n-1}, x_n) &\mapsto (x_1, \dots, x_{n-1}, x_n^{-1}) \text{ on } V_1 \\ ((x_1, \dots, x_{n-1}, x_n), t) &\mapsto (x_1, \dots, x_{n-1}, t x_n^{-1}) \text{ on } V_{01} \times \mathbb{A}^1, \end{aligned}$$

via the isomorphism $V_* \cong \mathbb{A}^n \setminus \{0\}$ described above.

We have the closed subschemes $\mathbb{A}_i^{n-1} \subset \mathbb{A}^n \setminus \{0\}$ defined by $x_i = 1$; the union (as presheaf) $\cup_{i=1}^n \mathbb{A}_i^{n-1}$ defines an \mathbb{A}^1 -contractible subpresheaf of $\mathbb{A}^n \setminus \{0\}$. Intersecting with V_0 , V_1 and V_{01} and forming the quotient sheaves \bar{V}_0 , \bar{V}_1 and \bar{V}_{01} we have the homotopy cocartesian diagram

$$\begin{array}{ccc} \bar{V}_{01} & \longrightarrow & \bar{V}_1 \\ \downarrow & & \downarrow \\ \bar{V}_0 & \longrightarrow & \mathbb{A}^n \setminus \{0\} / \cup_{i=1}^n \mathbb{A}_i^{n-1} \end{array}$$

One has $\bar{V}_{01} \cong (\mathbb{A}^{n-1} \setminus \{0\}, 1^{n-1}) \wedge \mathbb{G}_m$ and both \bar{V}_1 and \bar{V}_0 are contractible. Thus our automorphism ρ_n is given by the automorphism of the diagram

$$\bar{V}_{01} \rightrightarrows *$$

induced by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n^{-1})$ on V_{01} . Via the isomorphisms

$$\bar{V}_{01} \cong (\mathbb{A}^{n-1} \setminus \{0\}, 1^{n-1}) \wedge \mathbb{G}_m \cong \Sigma^{n-2} \wedge \mathbb{G}_m^{\wedge n-1} \wedge \mathbb{G}_m$$

we see that this is just the map $\Sigma^{n-2} \check{\sigma}_n$ and thus ρ_n is the suspension of this map, proving the result. \square

We present an alternate proof of Theorem 11.1 in the odd rank case, assuming the result in even rank. If $E \rightarrow X$ is an odd rank bundle, we consider the even rank bundle $V := p_1^* E \oplus p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ on $X \times \mathbb{P}^1$. By Lemma 5.7 V has Euler class

$$e(V) = p_1^* e(E) \cup p_2^* e(\mathcal{O}_{\mathbb{P}^1}(1)).$$

Let $i_0 : X \rightarrow X \times \mathbb{P}^1$ be the section to p_1 given by $i_0(x) = (x, 0)$. Since $p_2^*O_{\mathbb{P}^1}(1) \cong O_{X \times \mathbb{P}^1}(X \times 0)$, (5.2) gives the identity

$$p_2^*e(O_{\mathbb{P}^1}(1)) = i_{0*}(1_X),$$

hence

$$\begin{aligned} e(V) &= p_1^*e(E) \cup p_2^*e(O_{\mathbb{P}^1}(1)) \\ &= i_{0*}(i_0^*p_1^*e(E) \cup 1_X) \\ &= i_{0*}(e(E)). \end{aligned}$$

Since V has even rank, we have $e(V^\vee) = e(V)$, but by Corollary 8.6, $e(O_{\mathbb{P}^1}(1)^\vee) = (-\langle -1 \rangle)e(O_{\mathbb{P}^1}(1))$. Thus

$$\begin{aligned} e(V) &= e(V^\vee) = p_1^*e(E^\vee) \cup p_2^*e(O_{\mathbb{P}^1}(1)^\vee) \\ &= (-\langle -1 \rangle)i_{0*}(e(E^\vee)) \end{aligned}$$

by a computation similar to the one above. Applying p_{1*} we find

$$\begin{aligned} e(E) &= p_{1*}(i_{0*}(e(E))) \\ &= p_{1*}(e(V)) \\ &= p_{1*}(e(V^\vee)) \\ &= p_{1*}((-\langle -1 \rangle)i_{0*}(e(E^\vee))) \\ &= (-\langle -1 \rangle)e(E^\vee). \end{aligned}$$

We conclude this section with an application to the question of effectivity of the Euler characteristic.

Theorem 11.12. *Let X be a smooth projective variety of even dimension $2n$ over k . Suppose X admits a line bundle M such that, for $V = \Omega_{X/k}$ or $V = T_{X/k}$, the two conditions hold*

- (i) $V \otimes M^{\otimes 2}$ has a section s with isolated zero.
- (ii) $\deg_k(\sum_{i=1}^{2n} 2^{i-1}c_1(M)^i c_{2n-i}(V)) \leq 0$.

Then $\chi(X) \in \text{GW}(k)$ is effective, that is, $\chi(X)$ is represented by a quadratic form q over k .

Proof. Let V be a vector bundle of rank $2n$ on X . By Corollary 10.9 with $L = M^{\otimes 2}$ we have

$$e(V) = e(V \otimes L) - \bar{h} \cdot \left(\sum_{i=1}^{2n} 2^{i-1} c_1(M)^i c_{2n-i}(V) \right)$$

If $\det V = \omega_{X/k}^{\pm 1}$, this is an identity in $H^{2n}(X, \mathcal{K}_{2n}^{MW}(\omega_{X/k}))$. We may then push forward to $\text{Spec } k$. Letting

$$d = -\deg_k\left(\sum_{i=1}^{2n} 2^{i-1} c_1(M)^i c_{2n-i}(V)\right) \geq 0,$$

this gives

$$\pi_{X*}(e(V)) = \pi_{X*}(e(V \otimes L)) + d \cdot h.$$

If $V \otimes L$ admits a section s with isolated zeros, then by the work of Kass-Wickelgren [21], the Euler class with support $e_{s=0}(V \otimes L, s)$ is a sum of terms $e_x(V \otimes L, s)$, x a closed point in $s = 0$, with

$$e_x(V \otimes L, s) = i_{x*}([q_x])$$

where q_x is a quadratic form over $k(x)$. This implies that

$$\begin{aligned} \pi_{X*}(e(V)) &= \pi_{X*}(e(V \otimes L)) + d \cdot h \\ &= \sum_{x \in (s=0)} \pi_{x*}([q_x]) + d \cdot h \end{aligned}$$

which is the class of the quadratic form $q := d \cdot h + \sum_x \text{Tr}_{k(x)/k} q_x$ over k .

Since $e(\Omega_{X/k}) = e(T_{X/k})$ by Theorem 11.1, Theorem 1 implies

$$\chi(X) = \pi_{X*}(e(T_{X/k})) = \pi_{X*}(e(\Omega_{X/k})),$$

and we may apply these considerations to conclude that $\chi(X)$ is effective if the conditions (i) and (ii) hold for $V = T_{X/k}$ or $V = \Omega_{X/k}$. \square

Example 11.13. Let $i : X \hookrightarrow A$ be a smooth complete intersection defined by sections of semi-ample line bundles L_1, \dots, L_r on an abelian variety A such that at least one L_i is ample. Looking at the exact sequence

$$0 \rightarrow \bigoplus_{j=1}^r i^* L_j^{-1} \rightarrow i^* \Omega_{A/k} \rightarrow \Omega_{X/k} \rightarrow 0$$

we see that $\Omega_{X/k}$ is generated by global sections and a general section s has isolated zeros. The Euler class $\chi(X) \in \text{GW}(k)$ is therefore effective. As noted in Remark 1.11, if $k = \mathbb{R}$, this gives the inequality

$$|\chi(X(\mathbb{R})^{\text{an}})| \leq \chi(X(\mathbb{C})^{\text{an}}).$$

12. EULER CHARACTERISTICS AND FIBERING OVER A CURVE

We consider a projective morphism $f : Y \rightarrow X$, with Y a smooth projective integral k -scheme, and X a smooth projective curve over k . Kass and Wickelgren have raised the question of finding Grothendieck-Witt liftings of classical local-global type formulas for such maps and have obtained formulas of this type. We give a different approach here to this problem.

Due to the complicated behavior of the Euler classes with respect to tensor product with a line bundle, we make the following simplifying assumption

$$(12.1) \quad \text{There is a line bundle } M \text{ on } X \text{ with } \omega_{X/k} \cong M_0^{\otimes 2}.$$

This is the case if for instance $X = \mathbb{P}_k^1$ or if X is a hyperelliptic curve but is not satisfied if X is a conic without a rational point. The choice of a half-canonical line bundle M will not play an essential role.⁵

⁵A half-canonical line bundle is often referred to as a *theta characteristic*, see for example [8, 28].

Proposition 12.1. *Let f be a projective morphism $f : Y \rightarrow X$, with Y a smooth projective integral k -scheme of dimension r over k , and X a smooth projective curve over k . Suppose that X admits a half-canonical line bundle M . Then*

$$\begin{aligned} e(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}) &= e(\Omega_{Y/k}) \\ &+ \bar{h} \cdot c_1(f^* M^{-1}) \cup \left(\sum_{i=1}^r c_{r-i}(\Omega_{Y/k}) \cup c_1(f^* \omega_{X/k}^{-1})^{i-1} \right). \end{aligned}$$

Proof. This is just a special case of Corollary 10.9 \square

Under the assumption that X admits a half-canonical line bundle M , we may transform $e(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}) \in H^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k}^{-1} \otimes f^* \omega_{X/k}))$ to an element of $H^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k}))$ by applying the isomorphism $\psi_{\omega_{Y/k} \otimes M^{-1}}$. We will omit this comparison isomorphism from the notation in what follows.

Theorem 12.2. *Let f be a projective morphism $f : Y \rightarrow X$, with Y a smooth projective integral k -scheme of dimension r over k , and X a smooth projective curve over k . Let*

$$D(f) := (1/2)[\deg(c_r(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1})) - \deg(c_r(\Omega_{Y/k}))].$$

1. $D(f)$ is an integer.
2. Suppose X admits a half-canonical line bundle. Then

$$\pi_{Y*}(e(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1})) = (-\langle -1 \rangle)^r \chi(Y) + D(f) \cdot h$$

in $\text{GW}(k)$.

Proof. To compute the degree, we may assume that k is algebraically closed. Let $L = f^* \omega_{X/k}^{-1}$. Since $\omega_{X/k}$ has degree $2g_X - 2$, and k is algebraically closed, there is a half-canonical line bundle M on X . We have

$$c_r(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}) - \deg(c_r(\Omega_{Y/k})) = c_1(L) \cup \left(\sum_{i=1}^r c_{r-i}(\Omega_{Y/k}) \cup c_1(L)^{i-1} \right)$$

so

$$\begin{aligned} D(f) &= (1/2)[\deg[c_1(L) \cup \left(\sum_{i=1}^r c_{r-i}(\Omega_{Y/k}) \cup c_1(L)^{i-1} \right)]] \\ &= \deg[c_1(f^* M^{-1}) \cup \left(\sum_{i=1}^r c_{r-i}(\Omega_{Y/k}) \cup c_1(L)^{i-1} \right)] \end{aligned}$$

so $D(f)$ is an integer.

If X admits a half-canonical line bundle M (over the original field k), then by Proposition 12.1 and Theorem 11.1, we have

$$\begin{aligned}
& \pi_{Y*}(e(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1})) \\
&= \pi_{Y*}(e(\Omega_{Y/k})) + \bar{h} \cdot \pi_{Y*}(c_1(f^* M^{-1}) \cup (\sum_{i=1}^r c_{r-i}(\Omega_{Y/k}) \cup c_1(L)^{i-1})) \\
&= \pi_{Y*}(e(\Omega_{Y/k})) + D(f) \cdot h \\
&= (-\langle -1 \rangle)^r \chi(Y) + D(f) \cdot h.
\end{aligned}$$

□

We now turn to the discussion of the local invariants. As usual, a *critical point* of f is a point $y \in Y$ with $df(y) = 0$, a critical value of f is a point $x = f(y)$ of X with y a critical point. We assume that f has only finitely many critical points and let $\mathfrak{c}(f)$ denote the set of critical points.

Let M be the chosen half-canonical line bundle, and fix an isomorphism $\psi : M^{\otimes -2} \rightarrow \omega_{X/k}^{-1}$.

Let y be a critical point of f , giving the Euler class with support

$$e_y(df) := e_y(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df) \in H_y^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k}^{-1} \otimes f^* \omega_{X/k})).$$

Applying the comparison isomorphism $\psi_{\omega_{Y/k} \otimes M^{-1}}$ and the inverse of the Thom isomorphism

$$\mathcal{K}_0^{MW}(y) \cong H_y^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k}))$$

we will consider $e_y(df)$ as an element of $\mathcal{K}_0^{MW}(y)$. We have the pushforward

$$i_{y*} : \mathcal{K}_0^{MW}(y) \rightarrow H^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k})).$$

Corollary 12.3. *Let f be a projective morphism $f : Y \rightarrow X$, with Y a smooth projective integral k -scheme of dimension r over k , and X a smooth projective curve over k . Suppose X admits a half-canonical line bundle and that f has only finitely many critical points. Then*

$$\chi(Y) = D(f) \cdot h + (-\langle -1 \rangle)^r \sum_{y \in \mathfrak{c}(f)} \pi_{Y*} i_{y*} e_y(df).$$

in $\text{GW}(k)$.

Proof. Forgetting supports sends the Euler class with supports

$$e_{df=0}(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df) \in H_{df=0}^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k}^{-1}) \otimes f^* \omega_{X/k})$$

to $e(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df)$ in $H^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k}^{-1}) \otimes f^* \omega_{X/k})$. Applying the isomorphism $\psi_{\omega_{Y/k} \otimes M^{-1}}$ we have

$$\psi_{\omega_{Y/k} \otimes M^{-1}}(e(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df)) = \sum_{y \in \mathfrak{c}(f)} i_{y*} e_y(df).$$

Applying π_{Y*} and using Theorem 11.1, this gives

$$\chi(Y) = D(f) \cdot h + (-\langle -1 \rangle)^r \sum_{y \in \mathfrak{c}(f)} \pi_{Y*} i_{y*} e_y(df).$$

in $\text{GW}(k)$. \square

For a map f such that df has only isolated zeros, Kass and Wickelgren [21] have given an explicit formula for the local contributions $e_y(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df)$. We give here an explicit formula under the assumption that df is locally diagonalizable; this includes the case of non-degenerate critical points.

Let $y \in Y$ a critical points of f and let $x = f(y)$ be the corresponding critical value. Fix a half-canonical line bundle M , an isomorphism

$$\psi : M^{\otimes -2} \rightarrow \omega_{X/k}^{-1}$$

and a generating section $\lambda_{M,x}$ of M in a neighborhood of $x \in X$. A parameter $t_x \in \mathfrak{m}_x$ is *normalized* if

$$\partial/\partial t_x = \psi(\lambda_{M,x}^{-2}) \otimes k(x)$$

via the canonical isomorphism $\omega_{X/k}^{-1} \otimes k(x) \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.

Corollary 12.4. *Let f be a projective morphism $f : Y \rightarrow X$, with Y a smooth projective integral k -scheme of dimension r over k , and X a smooth projective curve over k . Suppose X admits a half-canonical line bundle. Suppose in addition that df is locally diagonalizable, with*

$$df_y \equiv \sum_{i=1}^r u_i^y t_i^{n_i^y} dt_i \otimes \partial/\partial t_x \pmod{\mathfrak{m}_y \cdot (t_1^{n_1}, \dots, t_d^{n_d})}$$

for each $y \in \mathfrak{c}(f)$. Here $t_1, \dots, t_n \in \mathfrak{m}_y$ is a system of parameters, $u_i^y \in \mathcal{O}_{Y,y}^\times$, $x = f(y)$ and $t_x \in \mathfrak{m}_x$ is a normalized parameter. Finally, suppose that $k(y)/k$ is a separable extension for each $y \in \mathfrak{c}(f)$. Let \bar{u}_i^y be the image of u_i^y in $k(y)^\times$. Then

$$\chi(Y) = D(f) \cdot h + (-\langle -1 \rangle)^r \sum_{y \in \mathfrak{c}(f)} \text{Tr}_{k(y)/k} \left(\left\langle \prod_{i=1}^d \bar{u}_i^y \right\rangle \cdot \prod_{i=1}^d (n_i^y)_\epsilon \right)$$

in $\text{GW}(k)$.

Proof. Under the isomorphism $\psi_{\omega_{Y/k} \otimes M}$ and the canonical isomorphism $\omega_{Y/k} \otimes \det^{-1} \mathfrak{m}_y/\mathfrak{m}_y^2 \cong k(y)$, the local contribution $e_y(df) := e_y(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df)$ is transformed to

$$\left(\left\langle \prod_{i=1}^r \bar{u}_i^y \right\rangle \cdot \prod_{i=1}^r (n_i^y)_\epsilon \right) \otimes \wedge dt_* \otimes \wedge \partial/\partial t_* = \left\langle \prod_{i=1}^r \bar{u}_i^y \right\rangle \cdot \prod_{i=1}^r (n_i^y)_\epsilon \in \mathcal{K}_0^{MW}(y).$$

Applying the pushforward

$$i_{y*} : \mathcal{K}_0^{MW}(y) \rightarrow H^r(Y, \mathcal{K}_r^{MW}(\omega_{Y/k})),$$

this gives

$$i_{y*}e_y(df) = i_{y*}(\langle \prod_{i=1}^r \bar{u}_i^y \rangle \cdot \prod_{i=1}^r (n_i^y)_\epsilon).$$

Thus

$$\begin{aligned} \pi_{Y*}(i_{y*}e_y(df)) &= \pi_{y*}(\langle \prod_{i=1}^r \bar{u}_i^y \rangle \cdot \prod_{i=1}^r (n_i^y)_\epsilon) \\ &= \mathrm{Tr}_{k(y)/k}(\langle \prod_{i=1}^d \bar{u}_i^y \rangle \cdot \prod_{i=1}^d (n_i^y)_\epsilon) \end{aligned}$$

whence

$$\begin{aligned} \chi(Y) &= D(f) \cdot h + (-\langle -1 \rangle)^r \sum_{y \in \mathfrak{c}(f)} \pi_{Y*}i_{y*}e_y(df) \\ &= D(f) \cdot h + (-\langle -1 \rangle)^r \sum_{y \in \mathfrak{c}(f)} \mathrm{Tr}_{k(y)/k}(\langle \prod_{i=1}^d \bar{u}_i^y \rangle \cdot \prod_{i=1}^d (n_i^y)_\epsilon). \end{aligned}$$

□

Let $f : Y \rightarrow X$ be as before a morphism of a smooth integral projective k -scheme Y of dimension r to a smooth projective curve X .

Let $y \in Y$ be a critical point of f . Let t_1, \dots, t_r be a system of parameters at y , and let $t_x \in \mathfrak{m}_x$ be a parameter. Since y is a critical point of f , $f^*(t_x)$ is in \mathfrak{m}_y^2 and thus

$$f^*(t_x) = \sum_{i \leq j} a_{ij} t_i t_j$$

for elements $a_{ij} \in \mathcal{O}_{Y,y}$, uniquely determined modulo \mathfrak{m}_y . Let $\bar{a}_{ij} \in k(y)$ be the residue of a_{ij} modulo \mathfrak{m}_y . Let

$$h_{ij} = \begin{cases} \bar{a}_{ij} & \text{if } i < j \\ 2\bar{a}_{ii} & \text{if } i = j \\ \bar{a}_{ji} & \text{if } i > j \end{cases}$$

The symmetric matrix

$$H(f)_y := (h_{ij})$$

is the *Hessian matrix* of f with respect to the chosen system of parameters. The point y is called a *non-degenerate critical point of f* if $H(f)_y$ is a non-singular matrix and $k(y)/k$ is a separable extension.

Let y be a non-degenerate critical point of f , let $x = f(y)$. Choose a system of parameters t_1, \dots, t_r at y and a parameter t_x at x , and let

$H(f)_y = (h_{ij})$ be the Hessian matrix of f . The section df_y satisfies

$$df_y \equiv \sum_{i,j} h_{ij} t_i dt_j \otimes \partial/\partial t_x \pmod{\mathfrak{m}_y^2}.$$

As $H(f)_y$ transforms as a symmetric bilinear form with respect to change of coordinates $(t_1, \dots, t_r) \mapsto (t'_1, \dots, t'_r)$, there is a system of parameters such that $H(f)_y$ is diagonal, in particular, df is locally diagonalizable near y . Furthermore, the local contribution in case $H(f)_y$ is diagonal is given by

$$e_y(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df) = \langle \det H(f)_y \rangle \otimes \partial/\partial t_x;$$

as $\det H(f)_y$ changes by a square in $k(y)^\times$ under change of system of parameters in \mathfrak{m}_y , this expression for $e_y(\Omega_{Y/k} \otimes f^* \omega_{X/k}^{-1}; df)$ is valid for any choice of parameters t_1, \dots, t_r in \mathfrak{m}_y .

Corollary 12.5. *Let f be a projective morphism $f : Y \rightarrow X$, with Y a smooth projective integral k -scheme and X a smooth projective curve over k . Suppose that X admits a half-canonical line bundle and that f has only non-degenerate critical points. For each $y \in \mathfrak{c}(f)$, let $x = f(y)$, choose a normalized parameter $t_x \in \mathfrak{m}_x$, and let $\langle \det H(f)_y \rangle \in \text{GW}(y)$ be the corresponding 1-dimensional quadratic form. Then*

$$\chi(Y) = D(f) \cdot h + (-\langle -1 \rangle)^r \sum_{y \in \mathfrak{c}(f)} \text{Tr}_{k(y)/k}(\langle \det H(f)_y \rangle)$$

in $\text{GW}(k)$.

Proof. This follows directly from Corollary 12.4 and the preceding discussion. \square

Remark 12.6. Suppose $X = \mathbb{P}_k^1$. Let $t = X_1/X_0$ be the standard parameter on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{(0 : 1)\}$. We have a unique isomorphism

$$\omega_{\mathbb{P}^1/k}^{-1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$$

sending $\partial/\partial t$ to X_0^2 and $\partial/\partial t^{-1}$ to $-X_1^2$. We use the section X_0 of $M := \mathcal{O}_{\mathbb{P}^1}(1)$ as our λ_M . For a closed point $x \in \mathbb{A}_k^1$, let $g_x \in k[t]$ be the monic irreducible polynomial for x over k . Then

$$t_x := (dg_x/dt)^{-1} g_x$$

is a normalized local parameter at x .

We apply these results to the case of a map of smooth projective curves $f : Y \rightarrow X$. For $p : C \rightarrow \text{Spec } k$ a smooth projective curve over k , define

$$g_{C/k} := \dim_k H^0(C, \omega_{C/k}).$$

This is the usual genus of C if C is geometrically integral over k .

Theorem 12.7 (Riemann-Hurwitz formula). *Let $f : Y \rightarrow X$ be a separable surjective morphism of smooth integral projective curves over k . Suppose that X admits a half-canonical bundle M and fix an isomorphism $\psi : M^{\otimes -2} \rightarrow \omega_{X/k}^{-1}$. For $y \in \mathfrak{c}(f)$, choose a parameter $t_y \in \mathfrak{m}_y$ and a normalized parameter $t_x \in \mathfrak{m}_x$, $x = f(y)$. Write*

$$f^*(t_x) = u_y t_y^{n_y}$$

with $u_y \in \mathcal{O}_{Y,y}^\times$ and let $\bar{u}_y \in k(y)^\times$ be the image of u_y . Suppose that $k(y)$ is separable over k and n_y is prime to $\text{char } k$ for all $y \in \mathfrak{c}(f)$. Then

$$\sum_{y \in \mathfrak{c}(f)} \text{Tr}_{k(y)/k}(\langle n_y \bar{u}_y \rangle (n_y - 1)_\epsilon) = (g_{Y/k} - 1 - \deg f \cdot (g_{X/k} - 1)) \cdot h.$$

Proof. Since f is separable and surjective, $\mathfrak{c}(f)$ is a finite set. Near $y \in \mathfrak{c}(f)$ we have

$$df = n_y u_y t_y^{n_y-1} \otimes dt_y \otimes \partial / \partial t_x.$$

By Corollary 12.4, we have

$$\sum_{y \in \mathfrak{c}(f)} \text{Tr}_{k(y)/k}(\langle n_y \bar{u}_y \rangle (n_y - 1)_\epsilon) = -\langle -1 \rangle \chi(Y) + D(f) \cdot h.$$

Since Y has odd dimension over k , $\chi(Y) = A \cdot h$ for some integer A , by Theorem 7.1. Thus

$$\sum_{y \in \mathfrak{c}(f)} \text{Tr}_{k(y)/k}(\langle n_y \bar{u}_y \rangle (n_y - 1)_\epsilon) = B \cdot h$$

for some integer B . Applying the rank homomorphism gives

$$\sum_{y \in \mathfrak{c}(f)} [k(y) : k](n_y - 1) = 2B$$

so the classical Riemann-Hurwitz formula tells us that

$$B = (g_{Y/k} - 1 - \deg f \cdot (g_{X/k} - 1)).$$

□

Remark 12.8. With notation as in Theorem thm:RH, suppose $y \in Y$ is a ramified point. Then

$$\langle n_y \bar{u}_y \rangle \cdot n_\epsilon = \begin{cases} \frac{1}{2}(n_y - 1) \cdot h & \text{if } n_y \text{ is odd,} \\ \langle n_y \bar{u}_y \rangle + \frac{1}{2}(n_y - 2) \cdot h & \text{if } n_y \text{ is even.} \end{cases}$$

We can rewrite the GW-Riemann-Hurwitz formula as

$$\begin{aligned} \sum_{y \in \mathfrak{c}(f), n_y \text{ even}} \text{Tr}_{k(y)/k}(\langle n_y \bar{u}_y \rangle) + \sum_{y \in \mathfrak{c}(f)} \left\lceil \frac{n_y - 1}{2} \right\rceil \cdot h \\ = (g_{Y/k} - 1 - \deg f \cdot (g_{X/k} - 1)) \cdot h. \end{aligned}$$

In other words, the ramification points with *odd* ramification indices impose a global relation in $\text{GW}(k)$ beyond the numerical identity given by the classical Riemann-Hurwitz formula.

Example 12.9. We take $k = \mathbb{R}$. Suppose we have a surjective map $f : Y \rightarrow \mathbb{P}_k^1$ with Y a smooth projective curve of genus g . Suppose in addition that f is simply ramified, that is, $n_y \leq 2$ for all $y \in Y$. Take a closed point $y \in Y$ with $n_y = 2$. If $k(y) = \mathbb{C}$, then the trace form $(e_i, e_j) \mapsto \text{Tr}_{\mathbb{C}/\mathbb{R}}(e_i e_j u)$ is hyperbolic for all $u \in \mathbb{C}^\times$. If $k(y) = \mathbb{R}$, then $\pi_{Y*}(e_y(df))$ is just the quadratic form $\langle 2\bar{u}_y \rangle$, using $t_x = t - f(y)$ as the normalized local parameter at $x = f(y)$ and writing $f^*(t_x) = u_y t_y^2$. Thus, the extra information in the GW-Riemann-Hurwitz formula is just that there are the same number of real ramified points y of f with $\bar{u}_y > 0$ as there are real ramified points y with $\bar{u}_y < 0$. This is also obvious by looking at the real points of Y , which is a disjoint union of circles, and using elementary Morse theory.

Remark 12.10. Going back to the guiding example of smooth projective varieties over \mathbb{R} , the formula of Corollary 12.5 for a map $f : Y \rightarrow \mathbb{P}^1$ may be viewed as combining the classical enumerative formulas for counting degeneracies for schemes over \mathbb{C} with using Morse theory to compute the Euler characteristic of a compact oriented manifold M by counting the number of critical points of a map $f : M \rightarrow S^1$ having only non-degenerate critical points, where we count a critical point with the sign of the Hessian determinant. In fact, as the signature of a hyperbolic form in $\text{GW}(\mathbb{R})$ is zero, and since the trace map $\text{Tr} : \text{GW}(\mathbb{C}) \rightarrow \text{GW}(\mathbb{R})$ sends q to $\text{rank}(q) \cdot h$, taking the signature of the formula in Corollary 12.5 expresses the Euler characteristic of $Y(\mathbb{R})^{\text{an}}$ as the sum of the signs of the Hessian determinant at each of the real critical points of f .

13. DIAGONAL HYPERSURFACES

We use Corollary 12.4 to compute the Euler characteristic of a diagonal hypersurface X in \mathbb{P}^{n+1} defined by $\sum_{i=0}^{n+1} a_i X_i^m$.

Fix an integer $m \geq 1$ and a base-field k of characteristic prime to $2m$; we may assume that k is infinite by replacing k with infinite extension of ℓ -power degree for some odd prime ℓ . Let $X = X(a_0, \dots, a_n; m) \subset \mathbb{P}^{n+1}$ be the hypersurface with defining equation $\sum_{i=0}^{n+1} a_i X_i^m = 0$, $a_i \in k^\times$. Let $\pi : \tilde{X} \rightarrow X$ be the blow-up along the closed subscheme Z defined by $X_n = X_{n+1} = 0$; note that $Z = X(a_0, \dots, a_{n-1}; m)$. The normal bundle $N_Z \rightarrow Z$ is thus $\mathcal{O}_Z(m)^2$, which is trivialized on $Z \setminus H$ for any hyperplane $H \subset \mathbb{P}^{n+1}$ not containing Z . Thus we may apply Proposition 1.10 to give

$$(13.1) \quad \chi(\tilde{X}) = \chi(X) + \langle -1 \rangle \chi(Z).$$

We have the morphism

$$f : \tilde{X} \rightarrow \mathbb{P}^1$$

induced by the rational map $X \rightarrow \mathbb{P}^1$, $(x_0 : \dots : x_{n+1}) \mapsto (x_n : x_{n+1})$. The map f has non-degenerate critical points $(0 : \dots : 0 : x_n : x_{n+1})$ satisfying $a_n x_n^m + a_{n+1} x_{n+1}^m = 0$ (the critical points do not lie over Z , so we may describe the critical points of f as points of X). Since $a_n a_{n+1} \neq 0$, the

critical points of f lie in the affine open subset $X_{n+1} \neq 0$, so we may use affine coordinates.

On the affine hypersurface $X^0 \subset \mathbb{A}^{n+1}$ defined by $\sum_{i=0}^n a_i x_i^m + a_{n+1} = 0$, the map f is given by

$$f(x_0, \dots, x_n) = x_n$$

and has critical subscheme $X_{crit} \subset X^0$ defined as a subscheme of \mathbb{A}^{n+1} by $x_i = 0$, $i = 0, \dots, n-1$, $x_n^m + a_{n+1}/a_n = 0$.

We apply the degeneration formula to the projection $f : \tilde{X} \rightarrow \mathbb{P}^1$. Let $y = X_{crit}$, a 0-dimensional reduced closed subscheme of X , and use the system of parameters (x_0, \dots, x_n) , generating the maximal ideal in $\mathcal{O}_{X, y'}$ for each closed point y' of y . Similarly, we let $g(T) = T^m + a_{n+1}/a_n$, let $x \subset \mathbb{A}_k^1$ be the subscheme $\text{Spec } k[T]/g(T)$ and use the parameter $t_x := (1/g'(T))g(T)$ in $\mathcal{O}_{\mathbb{A}^1, x}$.

As df is given by the expression

$$df = \sum_{i=0}^n (-1/a_n m x_n^{m-1}) \sum_{i=0}^n m a_i x_i^{m-1} dx_i \otimes \partial/\partial x_i \otimes \partial/\partial t_x$$

we see that df is locally diagonalizable and we have the local contribution

$$e_x(df) := \langle (-1/a_n x_n^{m-1})^n ((m-1)_\epsilon)^n \left(\prod_{i=0}^{n-1} a_i \right) \rangle \otimes \wedge dx_* \otimes \wedge \partial/\partial x_* \otimes \partial/\partial t_x$$

If n is odd, we know that $\chi(X)$ is hyperbolic, so we may assume that $n = 2r$ is even, in which case this expression reduces to

$$e_x(df) := \begin{cases} \frac{1}{2}(m-1)^n \cdot h \langle \left(\prod_{i=0}^{n-1} a_i \right) \rangle \otimes \wedge dx_* \otimes \wedge \partial/\partial x_* \otimes \partial/\partial t_x & \text{if } m \text{ is odd,} \\ \left[\frac{1}{2}((m-1)^n - 1) \cdot h + \langle 1 \rangle \right] \langle \left(\prod_{i=0}^{n-1} a_i \right) \rangle \otimes \wedge dx_* \otimes \wedge \partial/\partial x_* \otimes \partial/\partial t_x & \text{if } m \text{ is even.} \end{cases}$$

which reduces further to

$$e_x(df) := \begin{cases} \frac{1}{2}(m-1)^n \cdot h \otimes \wedge dx_* \otimes \wedge \partial/\partial x_* \otimes \partial/\partial t_x & \text{if } m \text{ is odd,} \\ \left[\left(\frac{1}{2}(m-1)^n - 1 \right) h + \left\langle \prod_{i=0}^{n-1} a_i \right\rangle \right] \otimes \wedge dx_* \otimes \wedge \partial/\partial x_* \otimes \partial/\partial t_x & \text{if } m \text{ is even.} \end{cases}$$

Following Remark 12.6, our choice of parameter t_x is normalized, so after applying the appropriate comparison isomorphism, we have the identity in

$$H^n(X, \mathcal{K}_n^{MW}(\omega_{X/k}))$$

$$i_{x*}(e_x(df)) = \begin{cases} i_{x*}[\frac{1}{2}(m-1)^n \cdot h] & \text{if } m \text{ is odd,} \\ i_{x*}[\frac{1}{2}((m-1)^n - 1)h + \langle \prod_{i=0}^{n-1} a_i \rangle] & \text{if } m \text{ is even,} \end{cases}$$

where

$$i_{x*} : H^0(x, \mathcal{K}_0^{MW}) \rightarrow H^n(X, \mathcal{K}_n^{MW}(\omega_{X/k}))$$

is the push-forward.

The extension $x/\text{Spec } k$ is a finite separable extension, so the trace form describes the pushforward map $p_* : \text{GW}(x) \rightarrow \text{GW}(k)$. Since

$$\text{Tr}_{k(x)/k}(\langle 1 \rangle) = \begin{cases} \langle m \rangle + \frac{m-1}{2} \cdot h & \text{for } m \text{ odd} \\ \langle m \rangle + \langle -ma_n a_{n+1} \rangle + \frac{m-2}{2} \cdot h & \text{for } m \text{ even,} \end{cases}$$

we get

$$p_{X*} i_{x*}(e_x(df)) = \begin{cases} \frac{1}{2}(m-1)^n m \cdot h; & \text{for } m \text{ odd,} \\ \langle m \prod_{i=0}^{n-1} a_i \rangle + \langle -m \prod_{i=0}^{n+1} a_i \rangle + \frac{1}{2}((m-1)^n - 1)m \cdot h; & \text{for } m \text{ even.} \end{cases}$$

Theorem 13.1. *Let $X = X(a_0, \dots, a_n; m) \subset \mathbb{P}_k^{n+1}$ be a diagonal hypersurface of degree $m \geq 1$. Suppose that $\text{char}(k) \nmid 2m$. Let $\delta(X) := \prod_{i=0}^n a_i$ and define $A_{n,m} \in \mathbb{Q}$ by*

$$A_{n,m} = \begin{cases} \frac{1}{2} \deg(c_n(T_X)) & \text{for } n \text{ odd,} \\ \frac{1}{2} (\deg(c_n(T_X)) - 1) & \text{for } n \text{ even and } m \text{ odd,} \\ \frac{1}{2} (\deg(c_n(T_X)) - 2) & \text{for } n \text{ and } m \text{ even.} \end{cases}$$

Then $A_{n,m}$ is an integer, depending only on n and m . Moreover,

$$\chi(X) = \begin{cases} A_{n,m} \cdot h & \text{for } n \text{ odd,} \\ A_{n,m} \cdot h + \langle m \rangle & \text{for } n \text{ even and } m \text{ odd,} \\ A_{n,m} \cdot h + \langle m \rangle + \langle -m\delta(X) \rangle & \text{for } n \text{ and } m \text{ even.} \end{cases}$$

Proof. It is clear that the rational number $A_{n,m}$ depends only on n and m . For n odd, the identity $\chi(X) = B \cdot h$ for some integer B follows from Theorem 7.1. Since $q(e(T_X)) = c_n(T_X)$ in $H^n(X, \mathcal{K}_n^M)$, we see that

$$2B = \text{rank}(\chi^{CW}(X)) = \deg(c_n(T_X)),$$

so $A_{n,m} = B$.

We now assume n is even and we prove the identity by induction on n . We first consider the case of even m . For $n = 0$,

$$X(a_0, a_1; m) = \text{Spec } k[T]/(T^m + a_{n+1}/a_n)$$

and $\chi(X)$ is given by the trace form,

$$\chi(X) = \mathrm{Tr}_{X/k}(\langle 1 \rangle) = \frac{m-2}{2} \cdot h + \langle m \rangle + \langle -m\delta(X) \rangle.$$

As $c_0(T_X)$ has degree m , the result is proven in this case. In general, assume the result for $n-2$, and let $Z = X(a_0, \dots, a_{n-1})$. Then combining (13.1), Corollary 12.4 and our computation of the local contributions (13.2), we have

$$\chi(X) = \langle m\delta(Z) \rangle + \langle -m\delta(X) \rangle - \langle -1 \rangle \chi(Z) + A \cdot h$$

for some integer A . Using our induction hypothesis, this reduces to

$$\chi(X) = -\langle -m \rangle + \langle -m\delta(X) \rangle + B \cdot h$$

for some integer B . But

$$h - \langle -m \rangle = \langle m \rangle + \langle -m \rangle - \langle -m \rangle = \langle m \rangle$$

so we have

$$\chi(X) = \langle m \rangle + \langle -m\delta(X) \rangle + (B-1) \cdot h.$$

In particular, this shows that $\deg(c_n(T_X)) = 2B$, which shows as above that $A_{n,m}$ is an integer and gives

$$\chi(X) = -\langle m \rangle + \langle -m\delta(X) \rangle + A_{n,m} \cdot h.$$

For odd m , the proof is essentially the same, starting with

$$\chi(X) = \mathrm{Tr}_{X/k}(\langle 1 \rangle) = \frac{m-1}{2} \cdot h + \langle m \rangle.$$

for $X = X(a_0, a_1; m)$; we leave the details to the reader. \square

For quadrics, this gives

Corollary 13.2. *Let k be a field with $\mathrm{char} k \neq 2$ and let Q be a non-singular quadric hypersurface in \mathbb{P}_k^{n+1} . Suppose Q has defining form q , with discriminant δ_q . Then*

$$\chi(Q) = \begin{cases} \frac{n+1}{2} \cdot h & \text{for } n \text{ odd,} \\ \frac{n}{2} \cdot h + \langle 2 \rangle + \langle -2\delta_q \rangle & \text{for } n \text{ even.} \end{cases}$$

This answers a question raised by Kass and Wickelgren.

Proof. If k is algebraically closed, then Q is cellular with $n+1$ cells in case n is odd, and $n+2$ cells if n is even. Thus

$$\mathrm{rank} \chi(Q) = \begin{cases} n+1 & \text{for } n \text{ odd,} \\ n+2 & \text{for } n \text{ even.} \end{cases}$$

With this, the corollary follows from Theorem 13.1, since every quadratic form is diagonalizable by a linear change of coordinates, and the discriminant is invariant modulo squares. \square

Remark 13.3. Applying Theorem 13.1 for $m = 1$ gives yet another proof that

$$\chi(\mathbb{P}^n) = \begin{cases} \frac{n+1}{2} \cdot h & \text{for } n \text{ odd,} \\ \langle 1 \rangle + \frac{n}{2} \cdot h & \text{for } n \text{ even.} \end{cases}$$

Remark 13.4. The fact that $\chi(X)$ depends only on m and n for m odd should not be surprising: every diagonal hypersurface $X(a_0, \dots, a_n; m)$ with m odd is isomorphic to $X(1, \dots, 1; m) \subset \mathbb{P}^{n+1}$ after a field extension of odd degree, and the base-extension map $\mathrm{GW}(k) \rightarrow \mathrm{GW}(F)$ for a finite field extension F/k is injective if $[F : k]$ is odd.

REFERENCES

- [1] H. Abelson, *On the Euler characteristic of real varieties*. Michigan Math. J. 23 (1976), no. 3, 267–271 (1977).
- [2] A. Asok, *Splitting vector bundles and \mathbb{A}^1 -fundamental groups of higher-dimensional varieties*, J. Topol. 6 (2013), no. 2, 311–348.
- [3] A. Asok, J. Fasel, *Splitting vector bundles outside the stable range and \mathbb{A}^1 -homotopy of punctured affine spaces*. J. Am. Math. Soc. Volume 28, Number 4, (2015) 1031–1062.
- [4] A. Asok, J. Fasel, *Comparing Euler classes*. Preprint 2013.
 \protect\vrule width0pt\protect\href{http://arxiv.org/abs/1306.5250}{arXiv:1306.5250}
 To appear Q. J. Math.
- [5] A. Asok, C. Haesemeyer, *The 0-th stable \mathbb{A}^1 -homotopy sheaf and quadratic zero cycles*. preprint 2011
 \protect\vrule width0pt\protect\href{http://arxiv.org/abs/1108.3854}{arXiv:1108.3854}.
- [6] A. Asok, M. Hoyois, M. Wendt, *Affine representability results in \mathbb{A}^1 -homotopy theory I: vector bundles*. preprint 2015
 \protect\vrule width0pt\protect\href{http://arxiv.org/abs/1506.07093}{arXiv:1506.07093}.
- [7] A. Asok, M. Hoyois, M. Wendt, *Affine representability results in \mathbb{A}^1 -homotopy theory II: principal bundles and homogeneous spaces*. preprint 2015
 \protect\vrule width0pt\protect\href{http://arxiv.org/abs/1507.08020}{arXiv:1507.08020}.
- [8] M. F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) 4 (1971) 47–62.
- [9] J. Ayoub, **Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I**. Astérisque No. 314 (2007), x+466 pp. (2008).
- [10] P. Balmer, *Witt groups*. Handbook of K-theory. Vol. 1, 2, 539–576, Springer, Berlin, 2005.
- [11] J. Barge, and F. Morel, *Groupe de Chow des cycles orientés et classe d’Euler des fibrés vectoriels*. C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 4, 287–290.
- [12] J. Fasel, *Groupes de Chow-Witt*. Mém. Soc. Math. Fr. (N.S.), 113 (2008).
- [13] J. Fasel, *The projective bundle theorem for I^j -cohomology*. J. K-Theory 11 (2013), no. 2, 413–464.
- [14] J. Fasel, V. Srinivas, *Chow-Witt groups and Grothendieck-Witt groups of regular schemes*. Advances in Mathematics 221 (2009) 302–329.
- [15] S. Gille, S. Scully and C. Zhong, *Milnor-Witt K-groups of local rings*. Adv. Math. 286 (2016), 729–753.
- [16] M. Hovey, *Model category structures on chain complexes of sheaves*. Trans. Amer. Math. Soc. 353 (2001), no. 6, 2441–2457.
- [17] M. Hoyois, *A quadratic refinement of the Grothendieck-Lefschetz-Verdier trace formula*. Algebr. Geom. Topol. 14 (2014) 3603–3658.

- [18] M. Hoyois, *The six operations in equivariant motivic homotopy theory*.
`\protect\vrule width0pt\protect\href{http://arxiv.org/abs/1509.02145}{arXiv:1509.02145}`
 to appear in Adv. Math.
- [19] P. Hu, *On the Picard group of the stable \mathbb{A}^1 -homotopy category*. Topology 44 (2005),
 no. 3, 609–640.
- [20] J. F. Jardine, *Motivic symmetric spectra*. Doc. Math. 5 (2000), 445–553.
- [21] J.L. Kass, K. Wickelgren, *The class of Eisenbud-Khimshiashvili-Levine is the local \mathbb{A}^1 -Brouwer degree* preprint 2016
`\protect\vrule width0pt\protect\href{http://arxiv.org/abs/1608.05669}{arXiv:1608.05669}`
 .
- [22] J. P. May, *The Additivity of Traces in Triangulated Categories*, Adv. Math. 163
 (2001), no. 1, pp. 34–73.
- [23] J. P. May, *Picard groups, Grothendieck rings, and Burnside rings of categories*. Adv.
 Math. 163 (2001), no. 1, 1–16.
- [24] J. Milnor, **Singular points of complex hypersurfaces**. Annals of Mathematics
 Studies, No. 61 Princeton University Press, Princeton, N.J.; University of Tokyo
 Press, Tokyo 1968.
- [25] F. Morel, *Introduction to \mathbb{A}^1 -homotopy theory*. Lectures given at the School on Alge-
 braic K-Theory and its Applications, ICTP, Trieste. 8-19 July, 2002.
- [26] F. Morel, **\mathbb{A}^1 -algebraic topology over a field**. Lecture Notes in Mathematics, 2052.
 Springer, Heidelberg, 2012.
- [27] F. Morel, V. Voevodsky, *\mathbb{A}^1 -homotopy theory of schemes*. Publ. Math. IHES 90 (1999)
 45–143.
- [28] D. Mumford, *Theta characteristics of an algebraic curve*, Ann. Sci. École Norm. Sup.
 (4) 4 (1971), 181–192.
- [29] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. 36 (1) (1976)
 167–171.
- [30] J. Riou, *Dualité de Spanier-Whitehead en géométrie algébrique*. C. R. Math. Acad.
 Sci. Paris 340 (2005), no. 6, 431–436.
- [31] C.A. Robinson, *Moore-Postnikov systems for non-simple fibrations*. Illinois J. Math.
 16 (1972), 234–242.
- [32] M. Schlichting, *Hermitian K-Theory of exact categories*. J. K-Theory 5 (2010), 105–
 165.
- [33] J.-P. Serre, *Espaces fibrés algébriques*, Exposé 5, Séminaire C. Chevalley, Anneaux de
 Chow et applications, 2nd années, IHP, 1958.
- [34] N. Spaltenstein, *Resolutions of unbounded complexes*. Compositio Math. 65 (1988),
 no. 2, 121–154.
- [35] Suslin, A. A. (1976), *Projective modules over polynomial rings are free*, Doklady
 Akademii Nauk SSSR, 229 (5) (1976) 1063–1066. Translated in Soviet Mathematics,
 17 (4) (1976) 1160–1164.
- [36] V. Voeodsky, *\mathbb{A}^1 -homotopy theory*. Proceedings of the International Congress of Math-
 ematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 579–604.
- [37] V. Voeodsky, *Motivic cohomology with $\mathbb{Z}/2$ -coefficients*. Publ. Math. Inst. Hautes
 Études Sci., (98):59–104, 2003.
- [38] C. Walter, *Grothendieck-Witt groups of triangulated categories* preprint (2003), K-
 theory server, <http://www.math.uiuc.edu/K-theory/643/TrigW.pdf>